Agents with state-dependent interactions

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(joint work with John Tsitsiklis and Julien Hendrickx, MIT)

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Lund University

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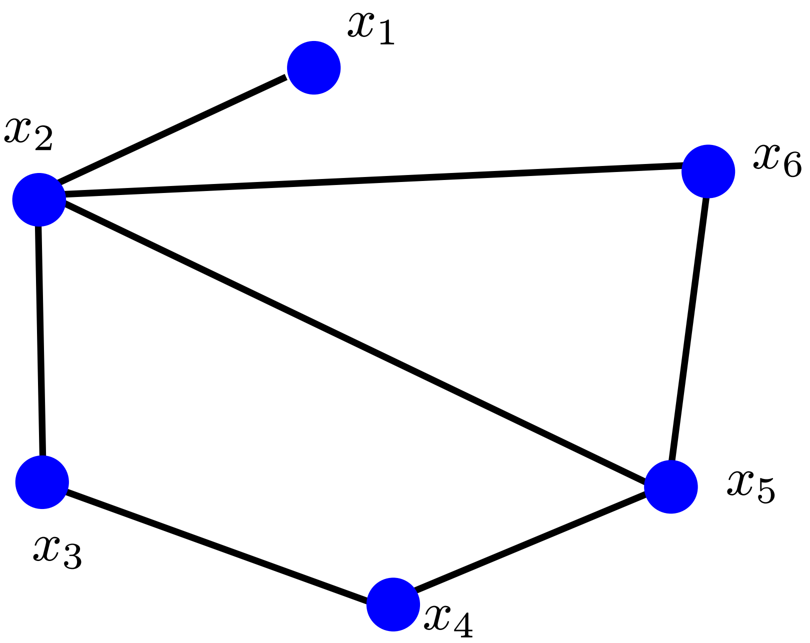
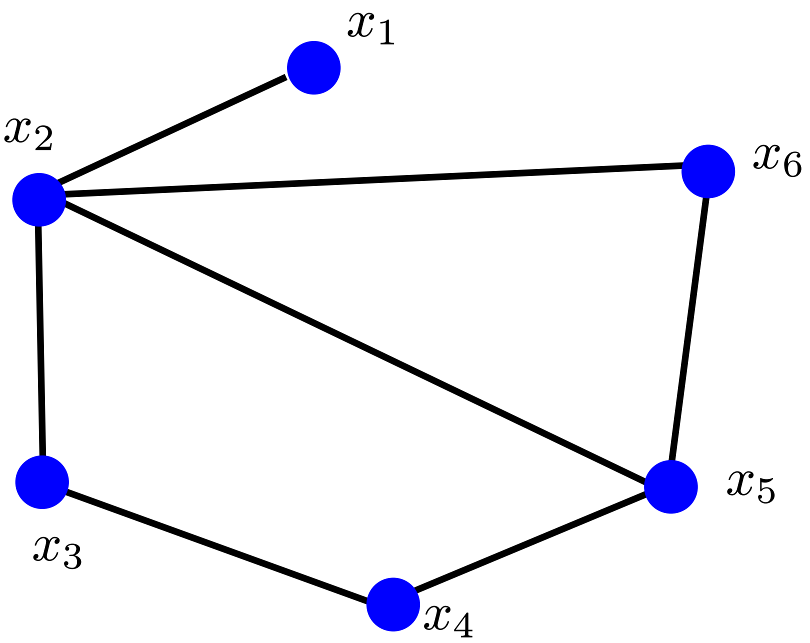
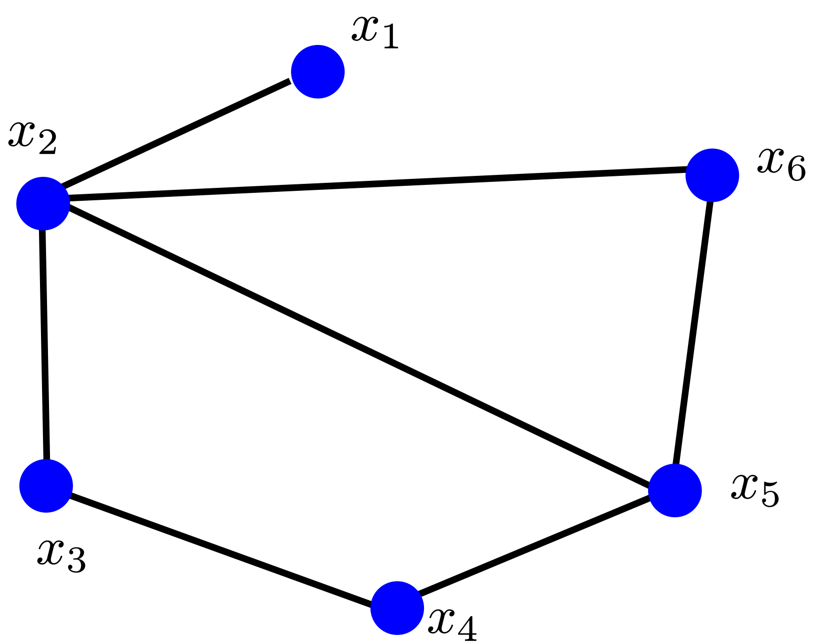


$$x_i(t+1) = \sum_{j:(i,j)\in E} a_{ij} x_j(t)$$

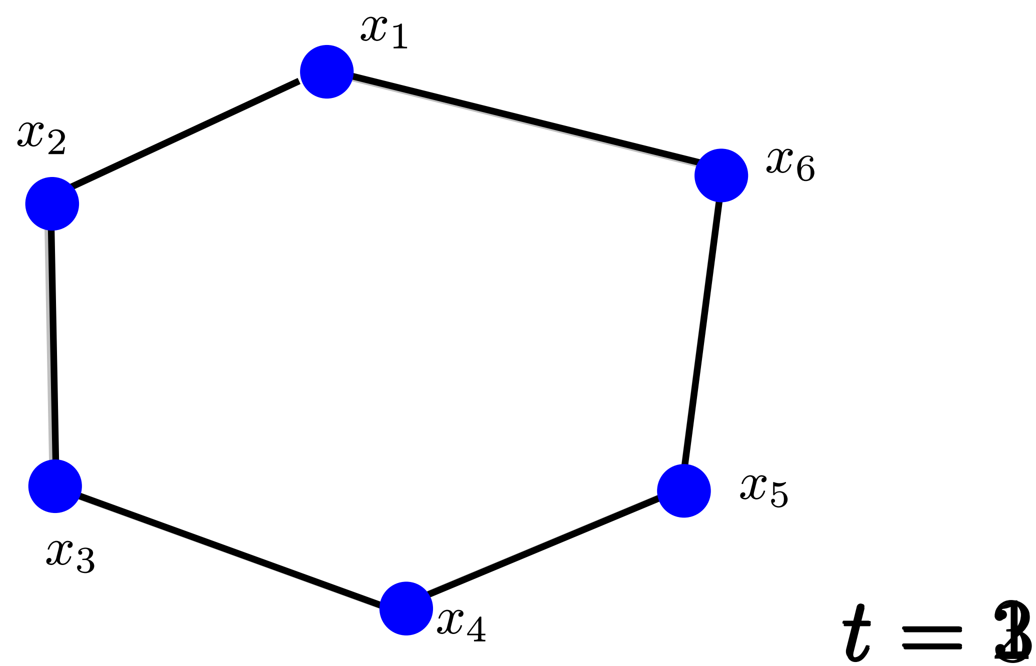


$$\dot{x}_i(t) = \sum_{j:(i,j)\in E} (x_j(t) - x_i(t))$$



Convergence? Consensus? Rate of convergence?

Graph of interconnection G: Fixed graph G or time-varying G(t)



Graph of interconnection G: Fixed graph G or time-varying G(t)

We consider state-dependent interconnection graphs G(x(t))

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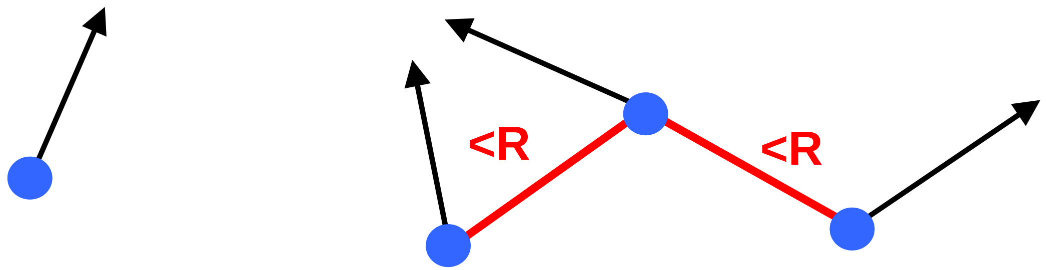
Vicsek's model (1995)

- Agents in the plane, same speed but different headings
- Neighbors if distant by less than R
- Headings updated by averaging neighbors headings.



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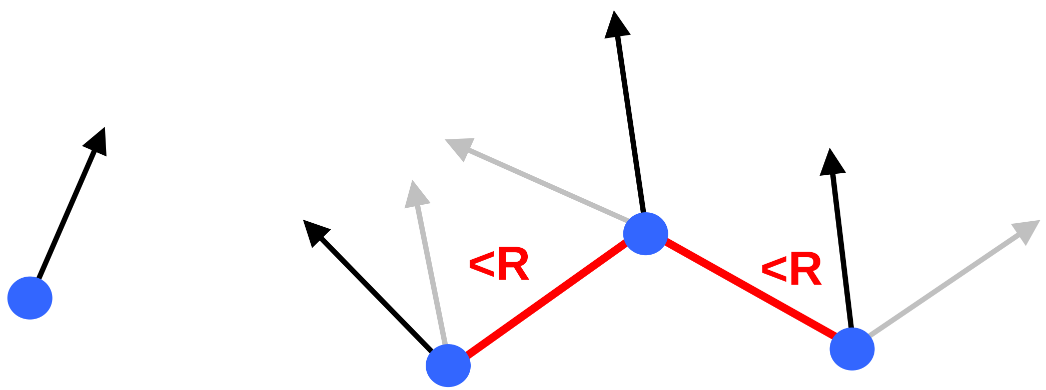

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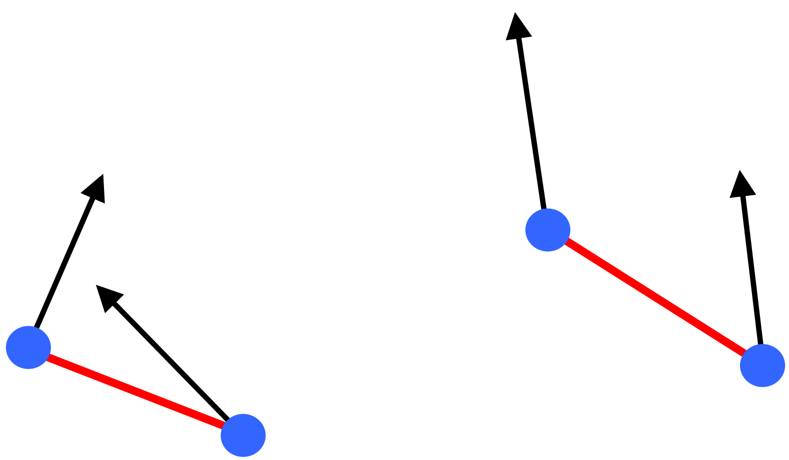
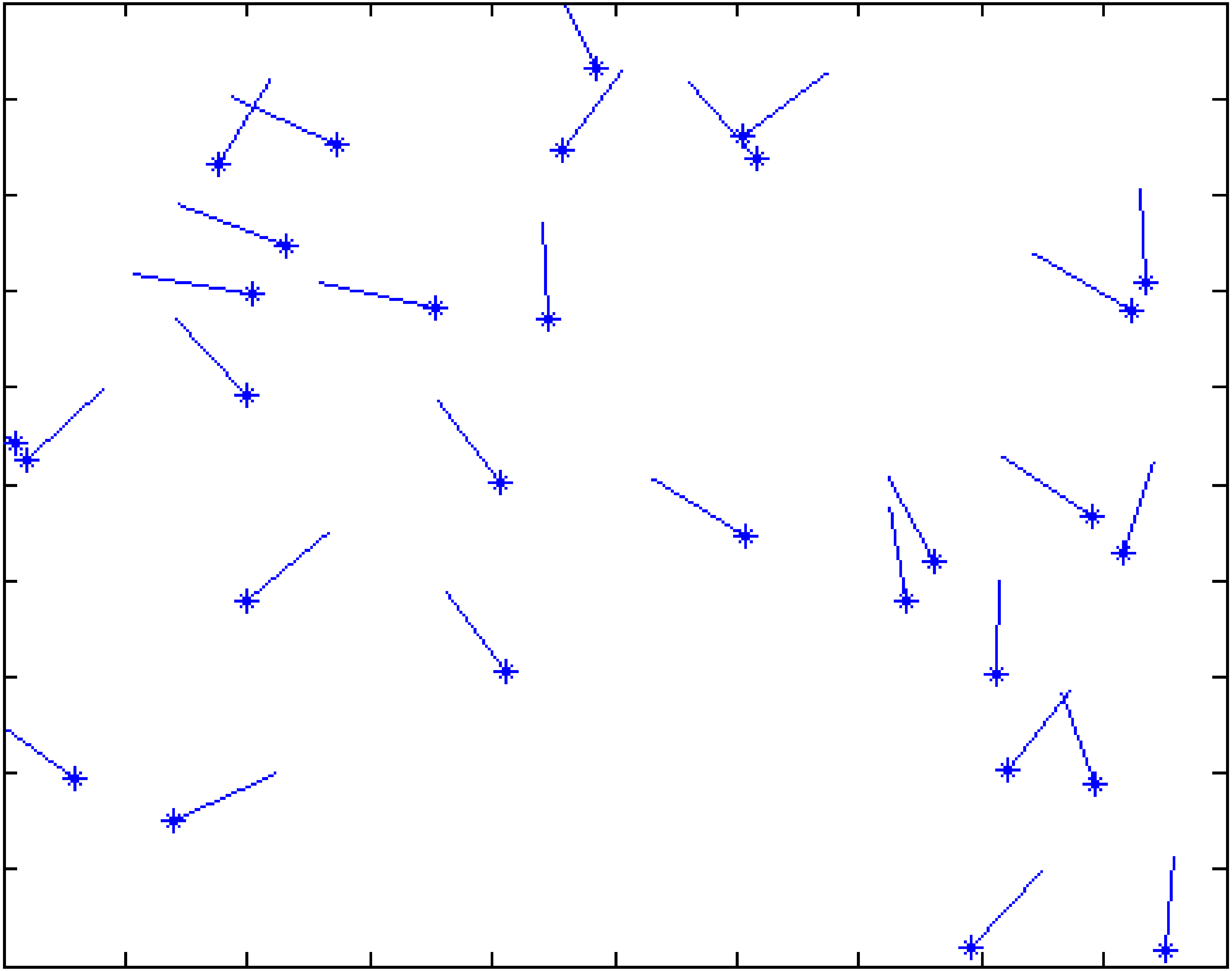
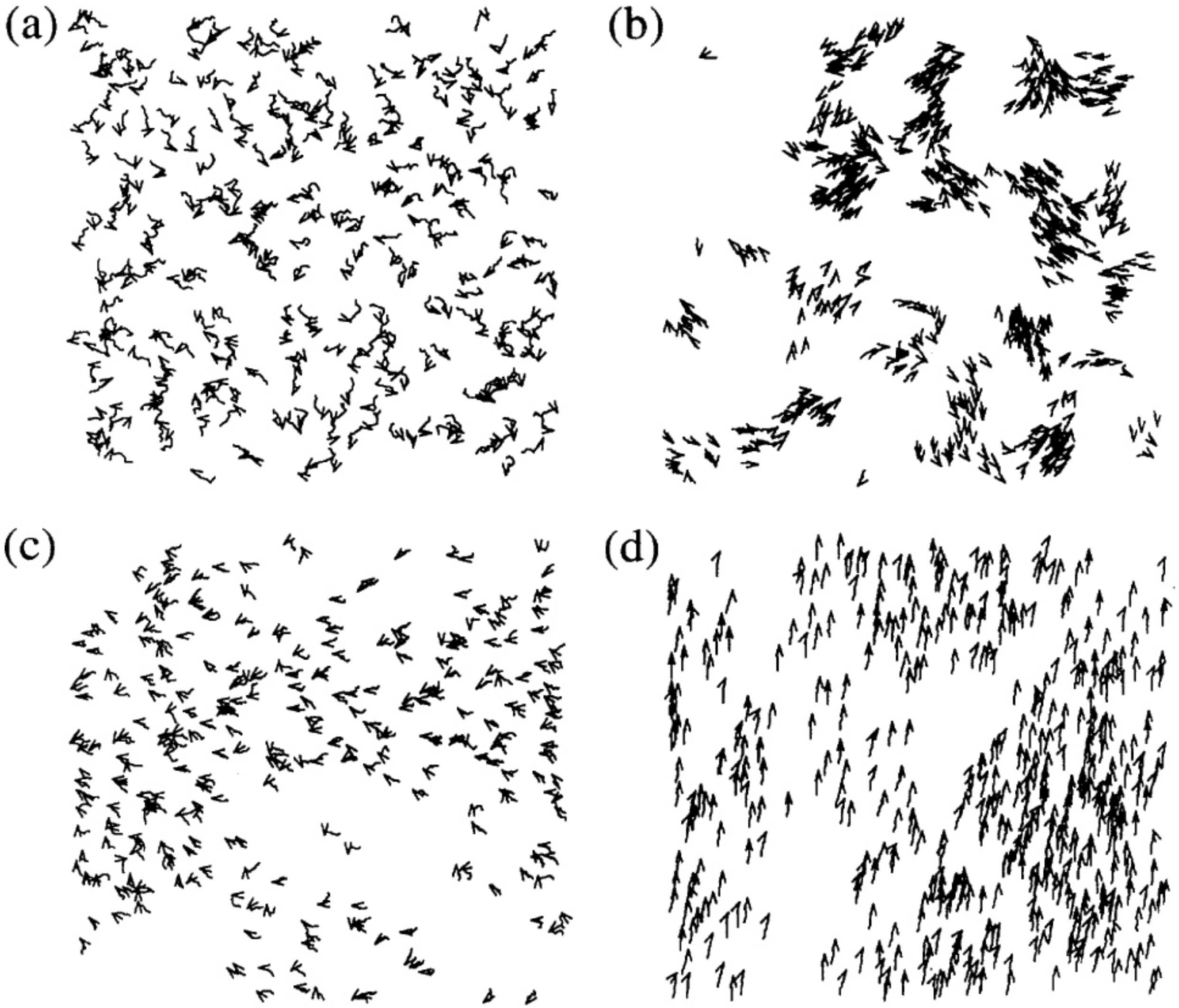




Novel Type of Phase Transition in a System of Self-Driven Particles

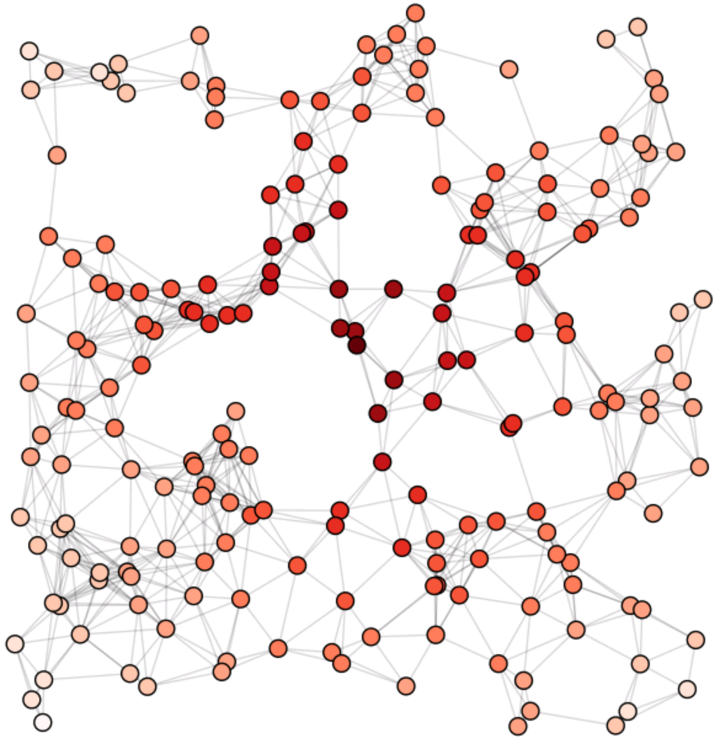
Tamás Vicsek,[1,2] András Czirók,[1] Eshel Ben-Jacob,[3] Inon Cohen,[3] and Ofer Shochet[3]

[1]*Department of Atomic Physics, Eötvös University, Budapest, Puskin u 5-7, 1088 Hungary*
[2]*Institute for Technical Physics, Budapest, P.O.B. 76, 1325 Hungary*
[3]*School of Physics, Tel-Aviv University, 69978 Tel-Aviv, Israel*
(Received 25 April 1994)

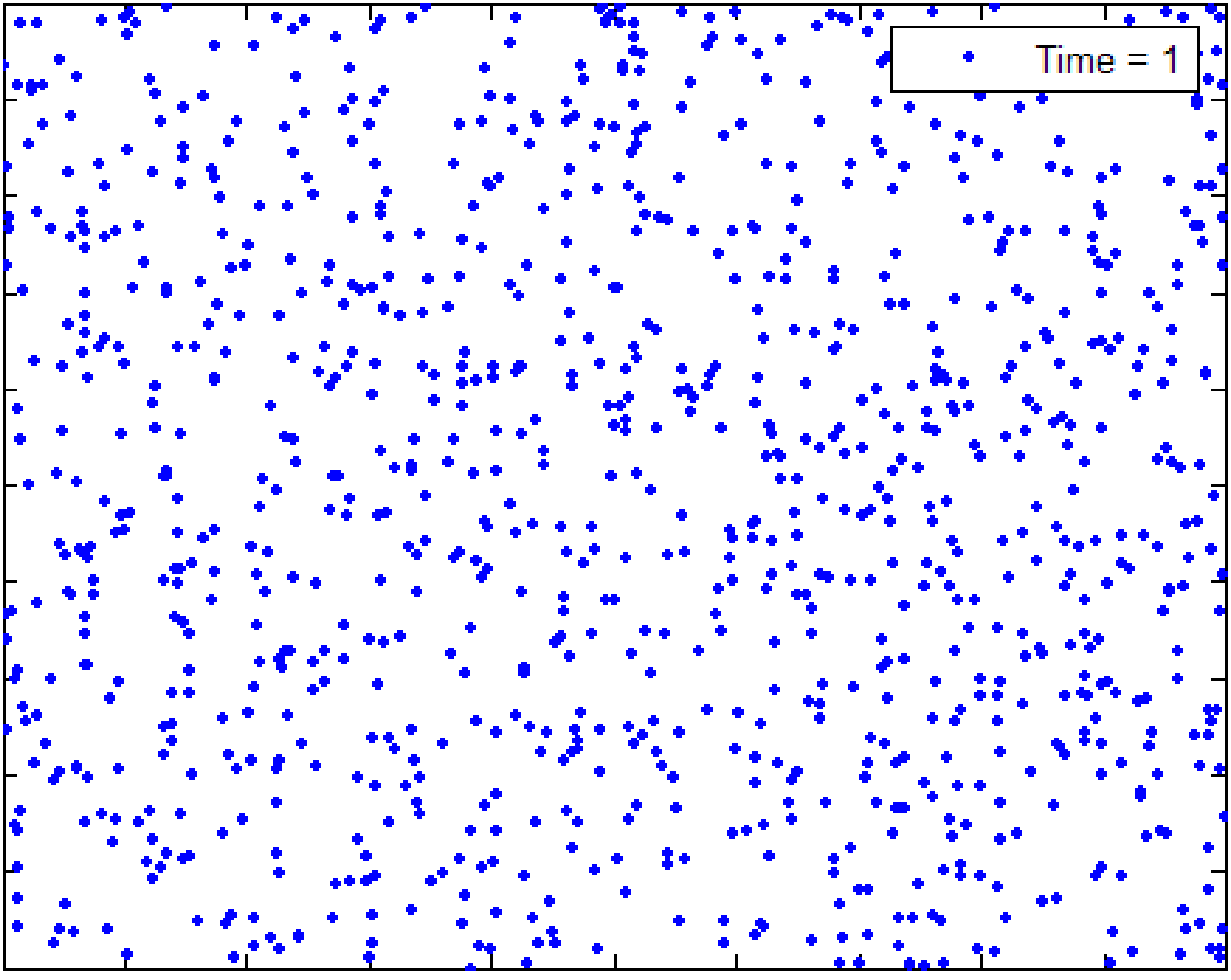


Krause’s model (2002)

Agents in the plane move to the center of mass of those at distance < R.



Geometric graph


Time = 1

Krause's model in 1D
Opinion dynamics

N agents each having a real "value/opinion", synchronously updated by averaging other opinions distant by less than R=1

Mixing beliefs among interacting agents

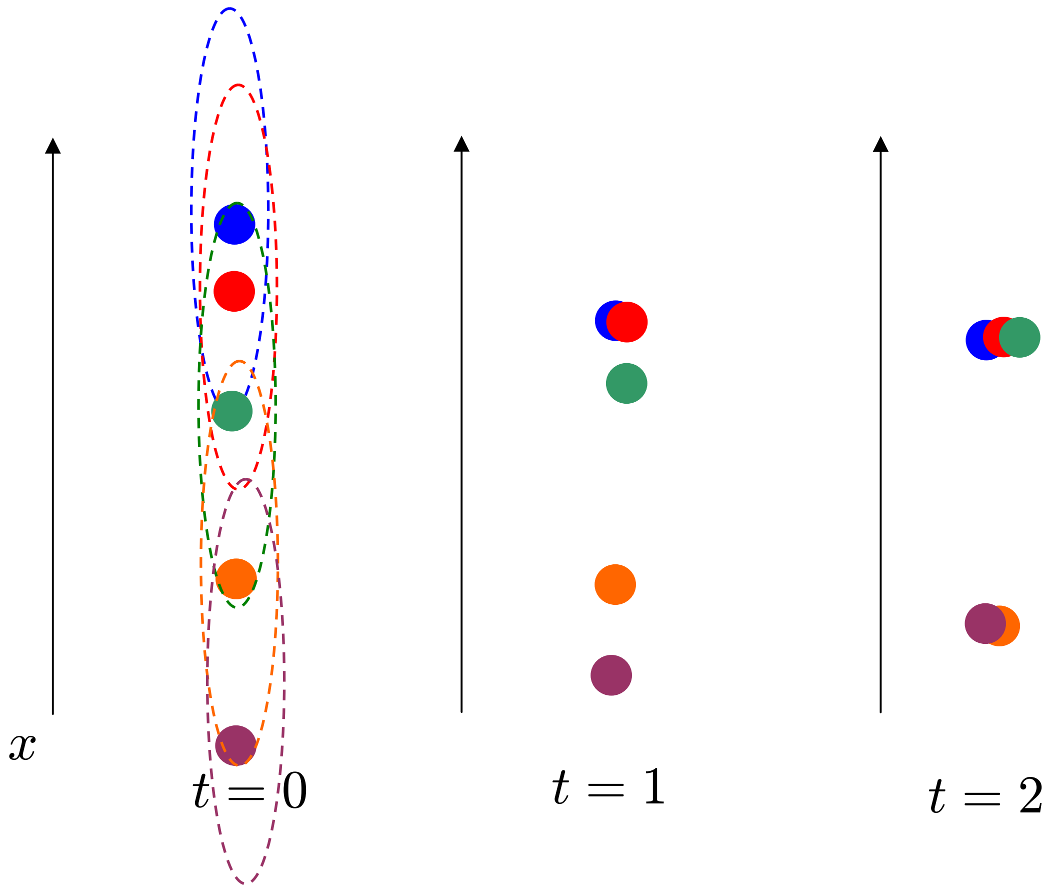
Advances in Complex Systems, 2001

Guillaume Deffuant*, David Neau, Frederic Amblard***
and Gérard Weisbuch**

The rationale for the threshold condition is that agents only interact when their opinion are already close enough; otherwise they do not even bother to discuss. The reason for such behaviour might be for instance lack of understanding, conflicts of interest or social pressure. Although there is no reason to suppose that openness to

Krause's model in 1D
Opinion dynamics

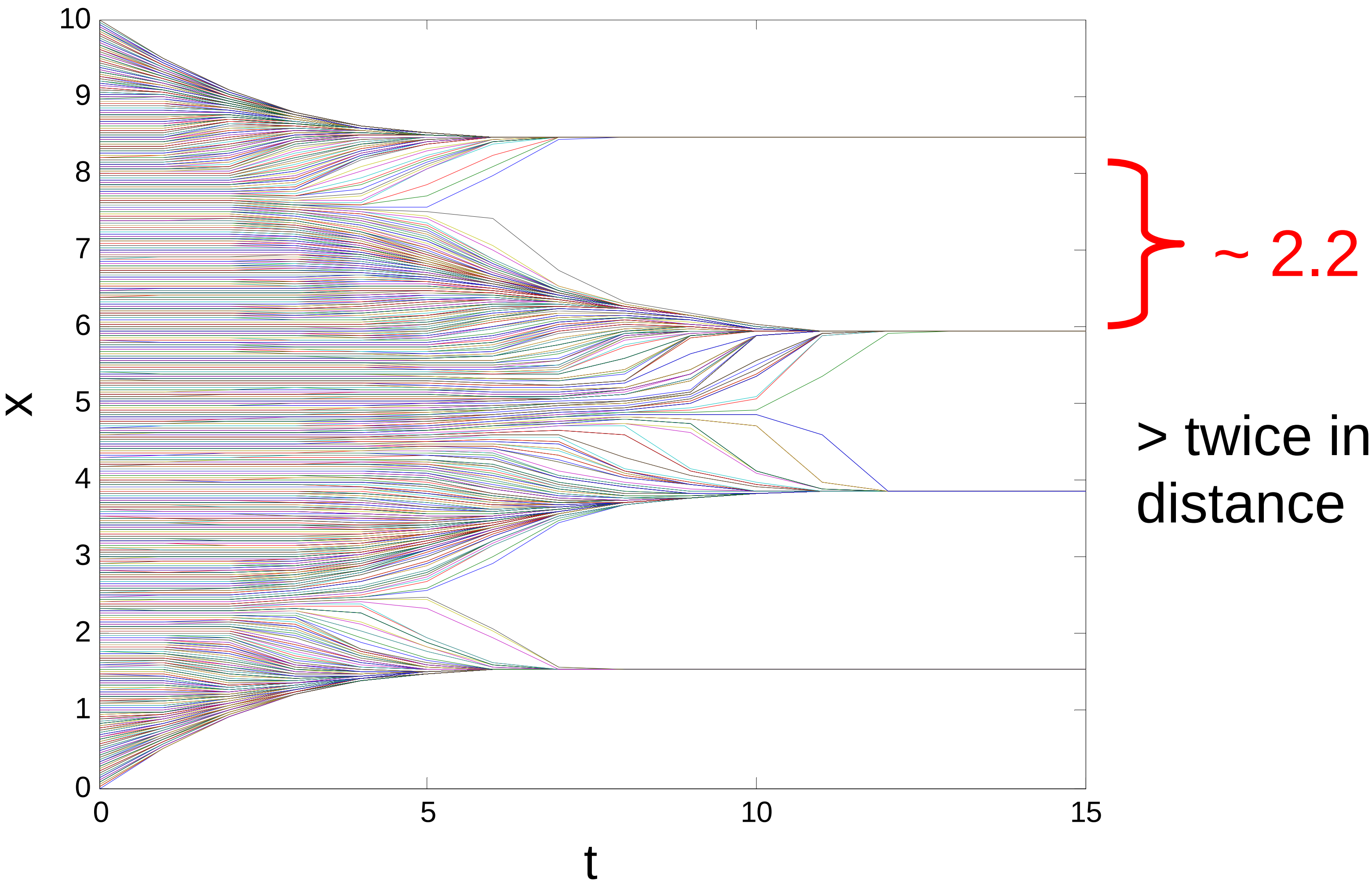
N agents each having a real "value/opinion", synchronously updated by averaging other opinions distant by less than R=1



Convergence?

Continuous time version

$$\dot{x}_i(t) = \sum_{j:|x_i(t)-x_j(t)|<1} (x_j(t) - x_i(t))$$

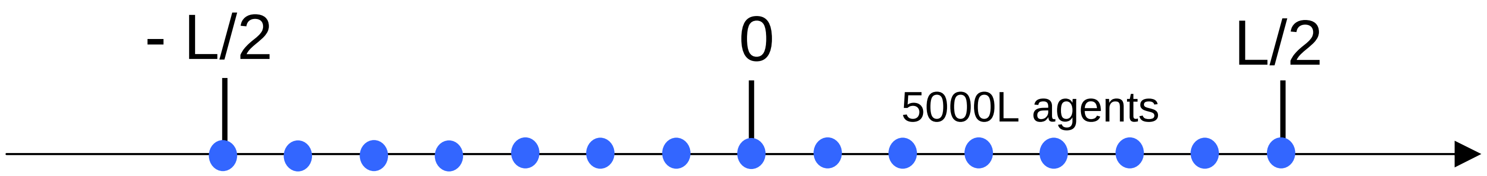


> twice interaction distance

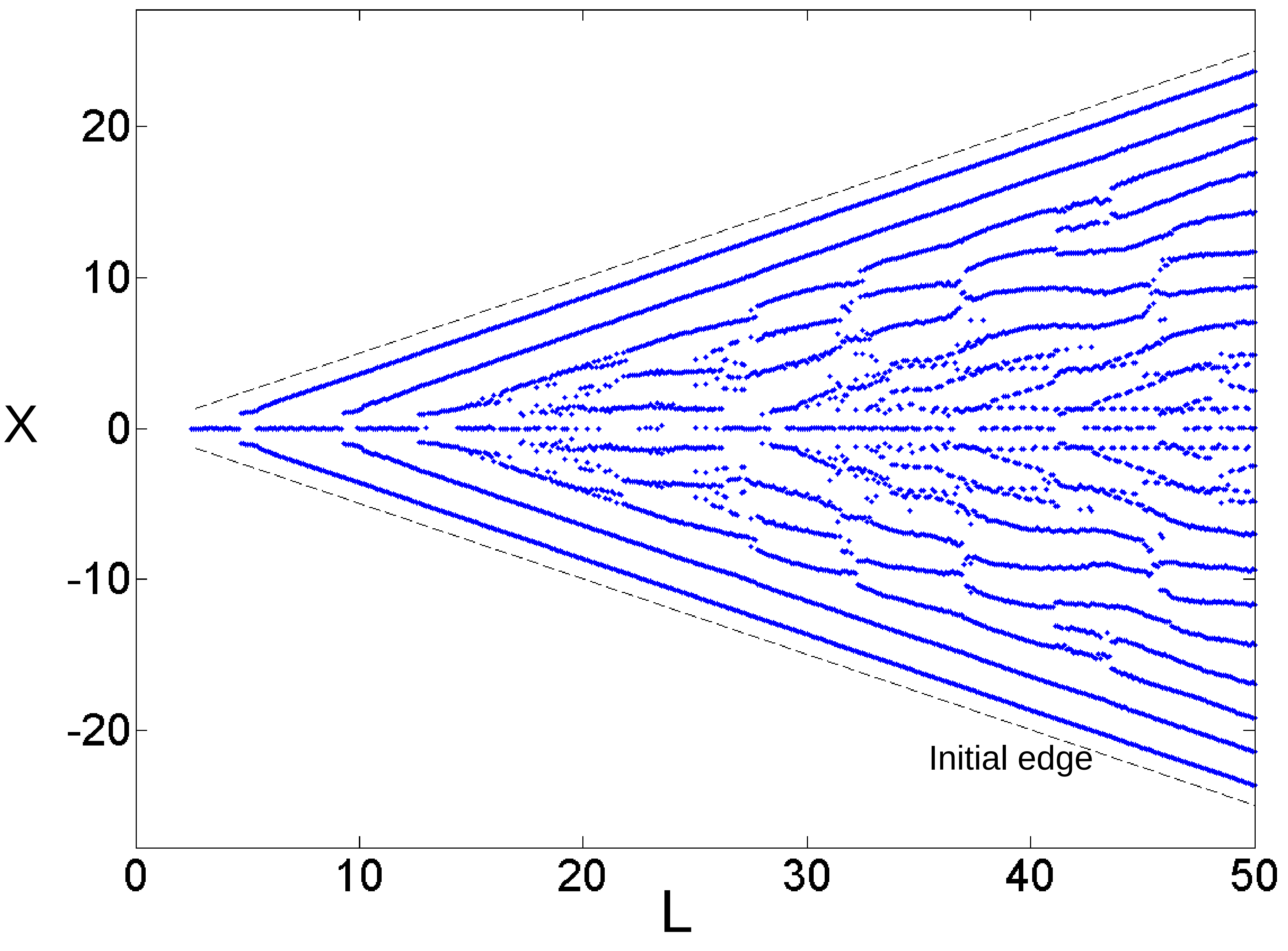
Initial opinions equidistantly distributed on [0, 10]

Number of clusters?

Equidistant initial distribution on an interval [-L/2,L/2]



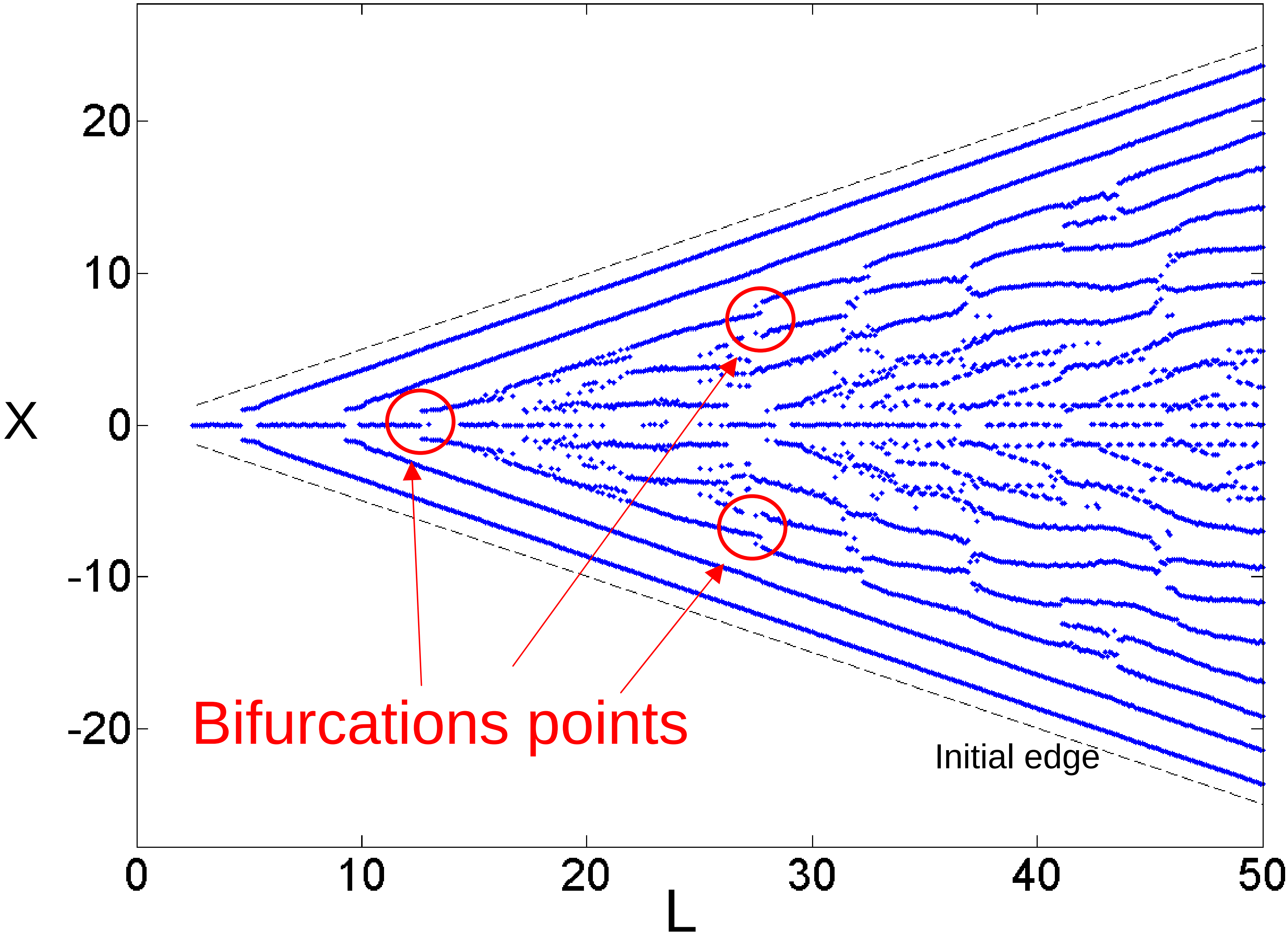
Incremental values of L


X
20
10
0
-10
-20
0
10
20
30
40
50
L
Initial edge

Distance between clusters ~ 2.2

WHY ??

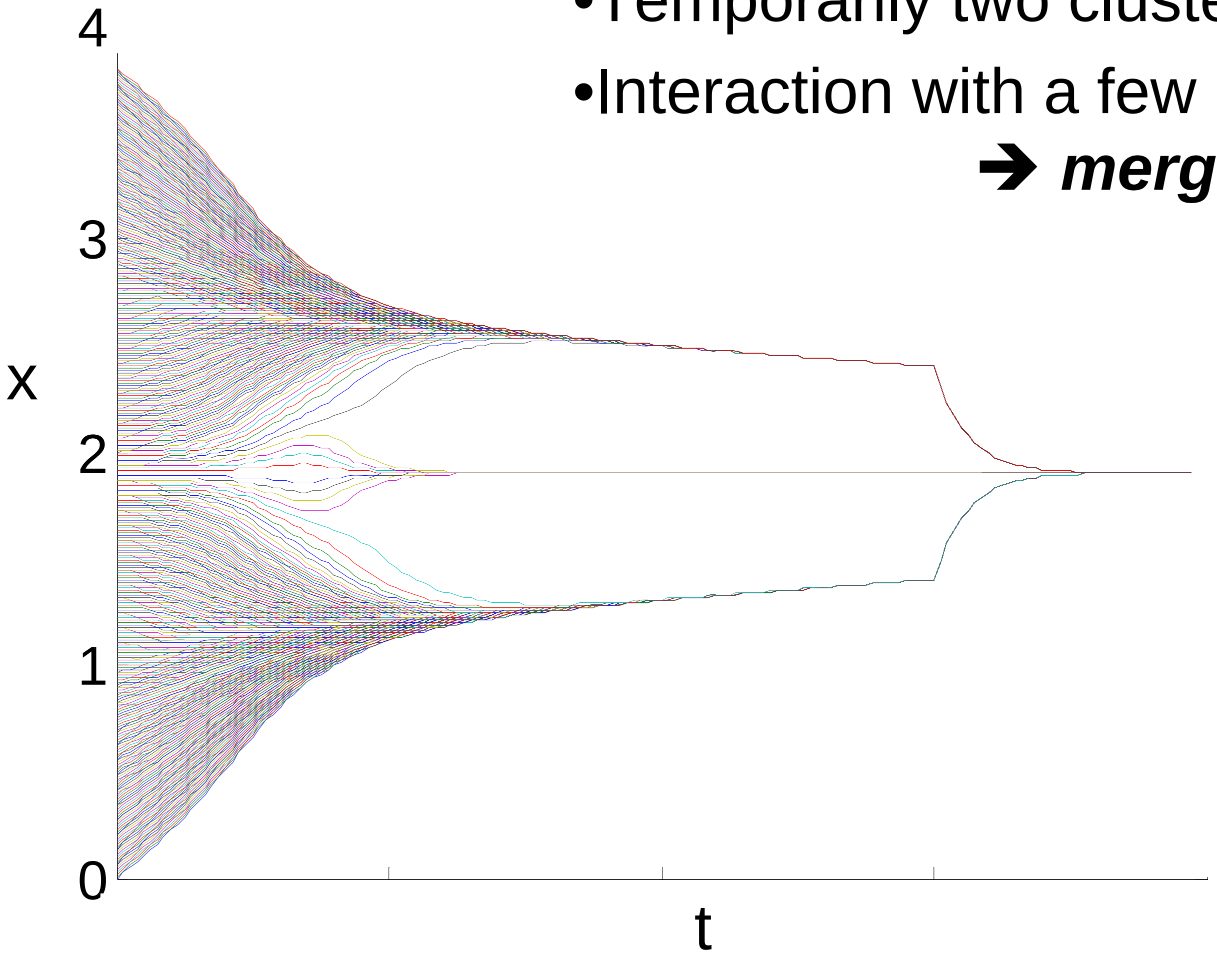
Equilibriums with distance ~ 2.2 more stable ?



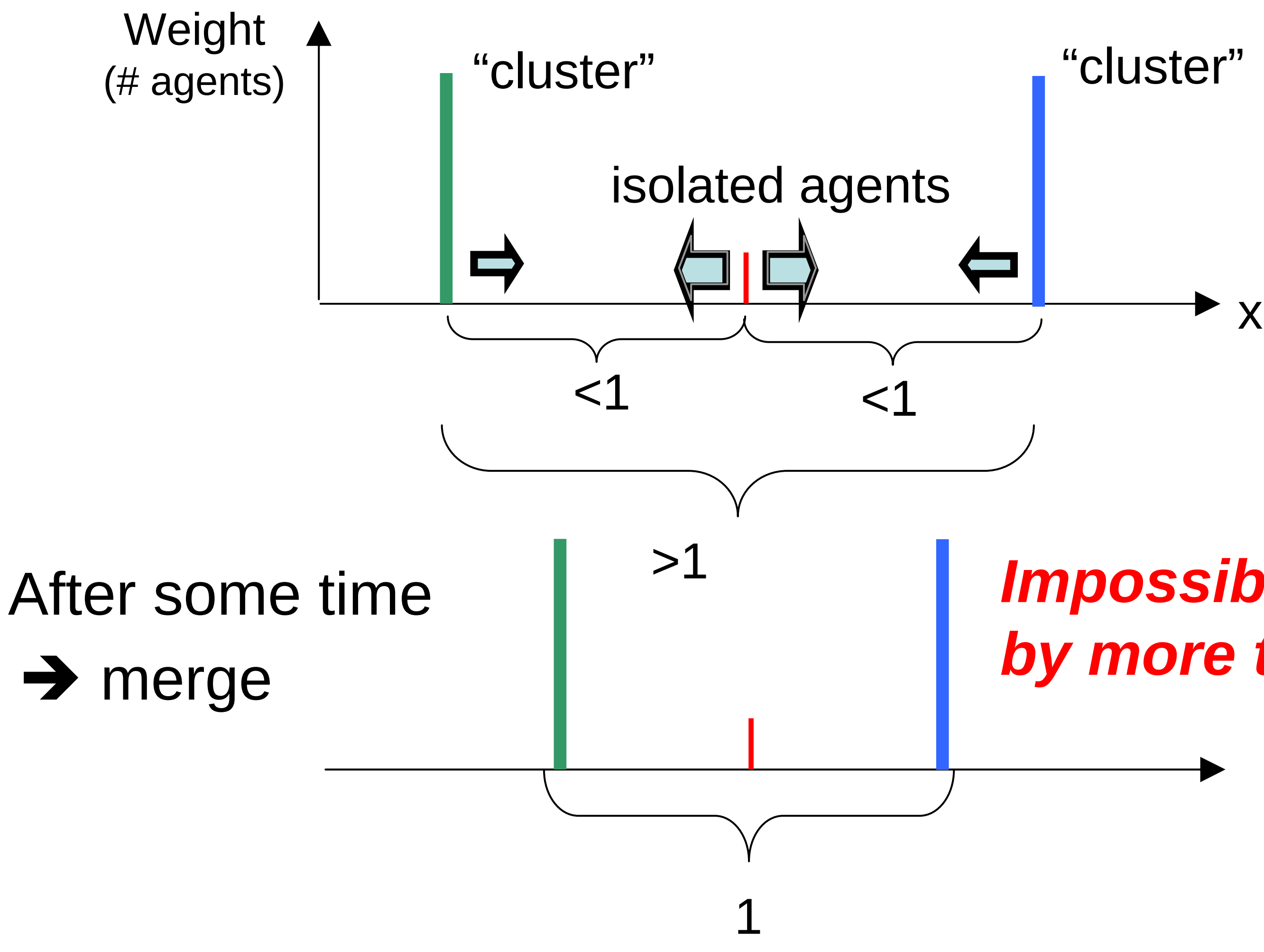
20
10
0
-10
-20
X
0
10
20
30
40
50
L
Bifurcations points
Initial edge

Equilibrium destroyed by a few agents

- Temporarily two clusters
- Interaction with a few isolated agents
  ➔ ***merge***



Equilibrium destroyed by a few agents



Impossible if separated by more than 2

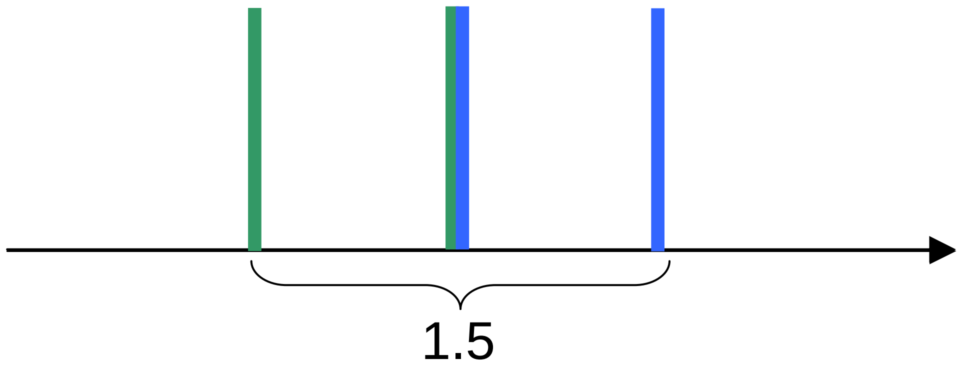
Stability multi-agent equilibrium

$$\dot{x}_i(t) = \sum_{j:|x_i(t)-x_j(t)|<1} w_j \left(x_j(t) - x_i(t)\right)$$

Equilibrium stable if largest perturbation resulting from addition of agent of small weight is small

Stability multi-agent equilibrium

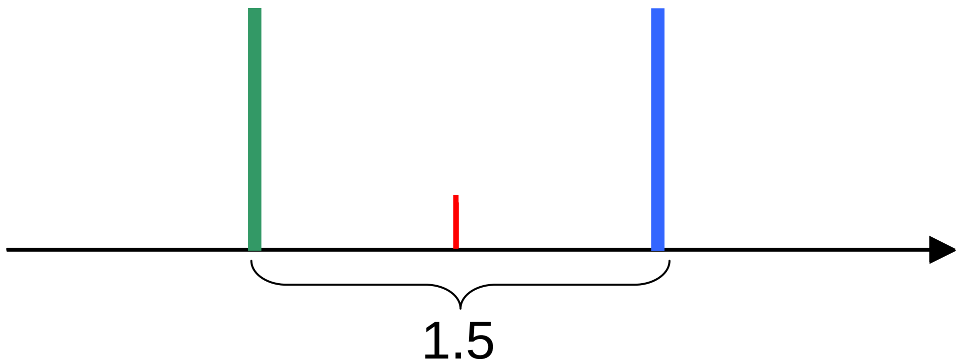
Unstable:



Addition of arbitrary small agent between clusters results in a merge

Stability multi-agent equilibrium

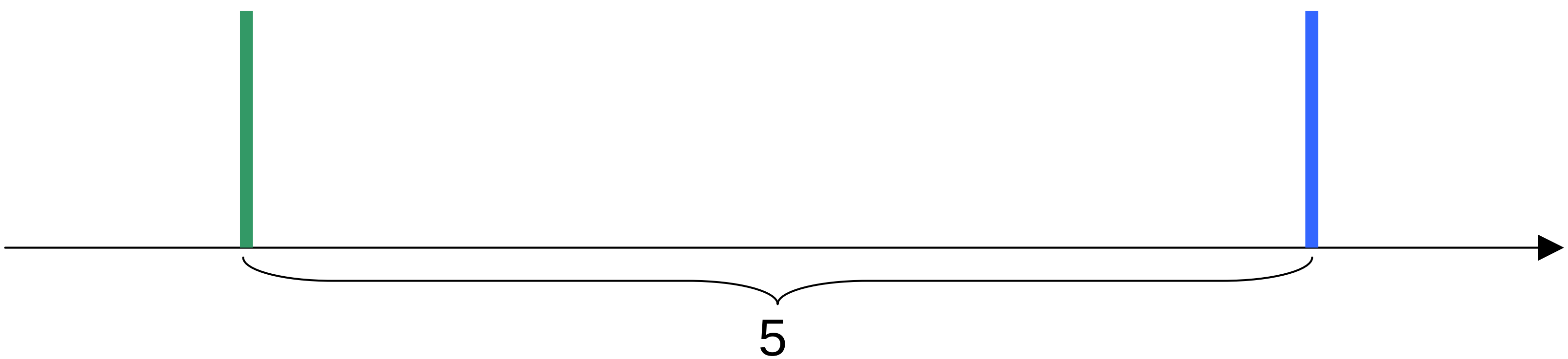
Unstable:



Addition of arbitrary small agent between clusters results in a merge

Stable:

Deformation < O(ε)



Inter-cluster distance for stability

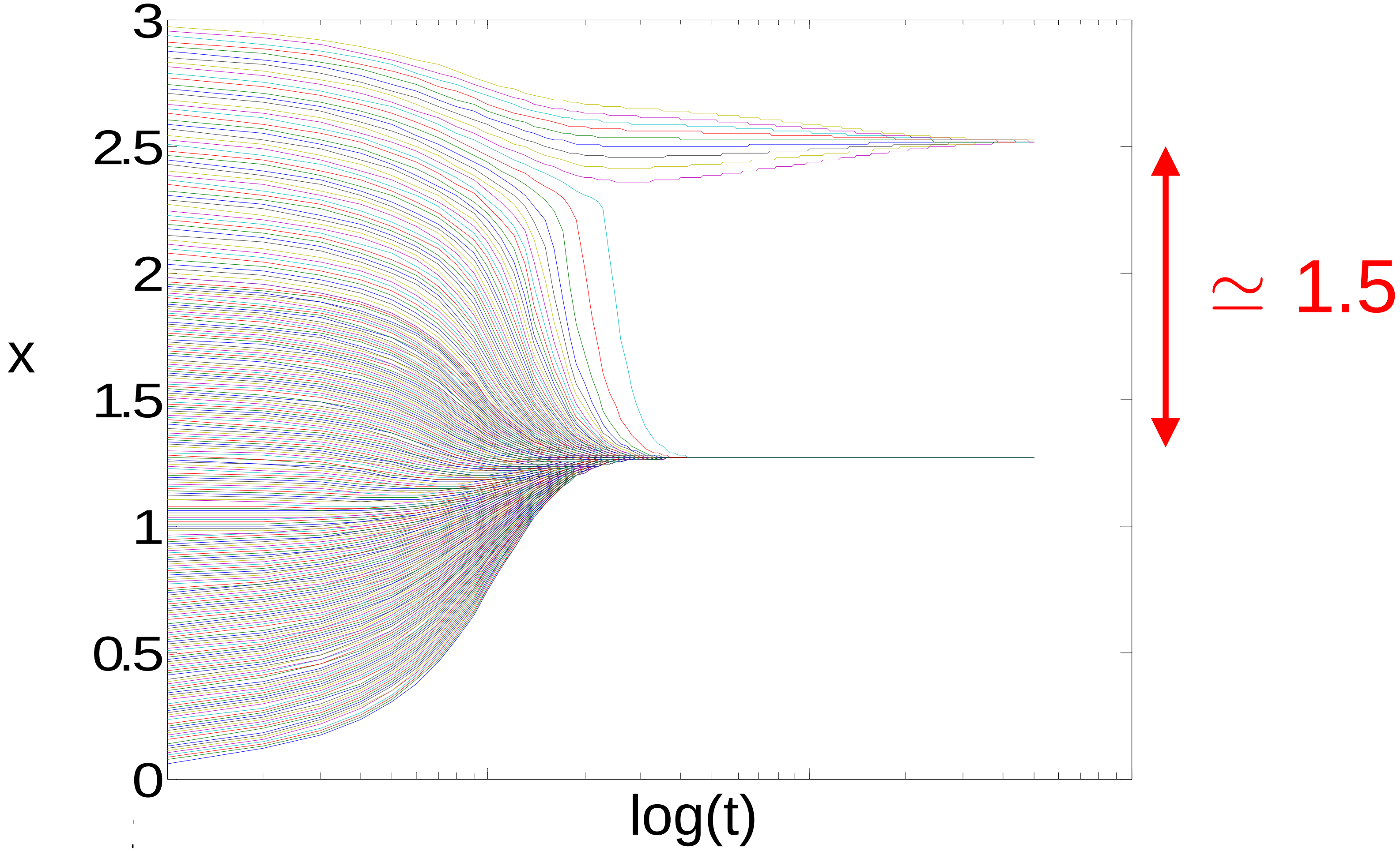
Theorem: An equilibrium is stable **if and only if** every two clusters A, B are separated by more than

$$1 + \frac{\min(W_A, W_B)}{\max(W_A, W_B)}$$

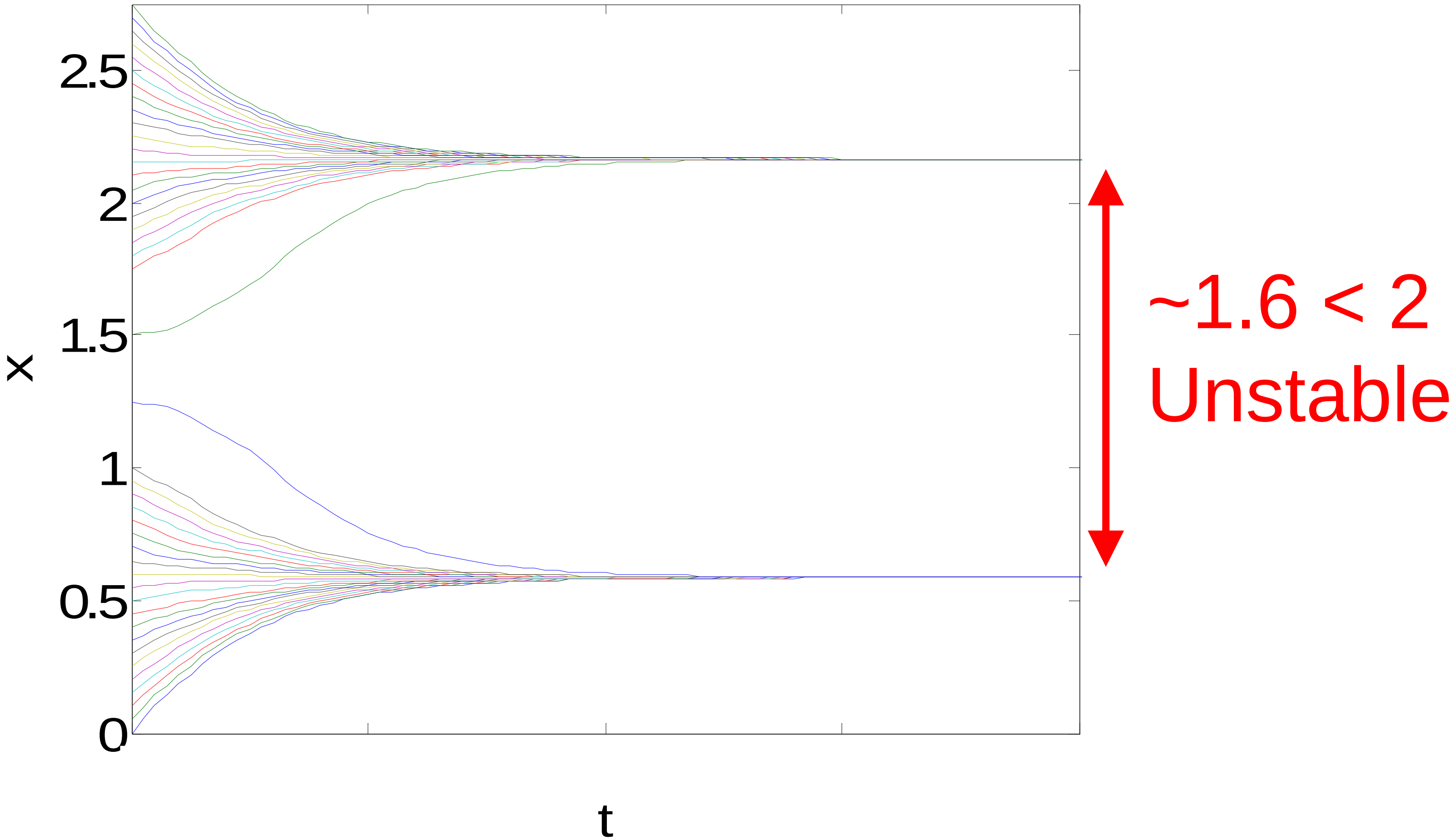
W_A, W_B : cluster weights

Corollary: At a stable equilibrium with $W_A = W_B$, the inter-cluster distance must be at least 2.

Convergence to unstable equilibrium possible

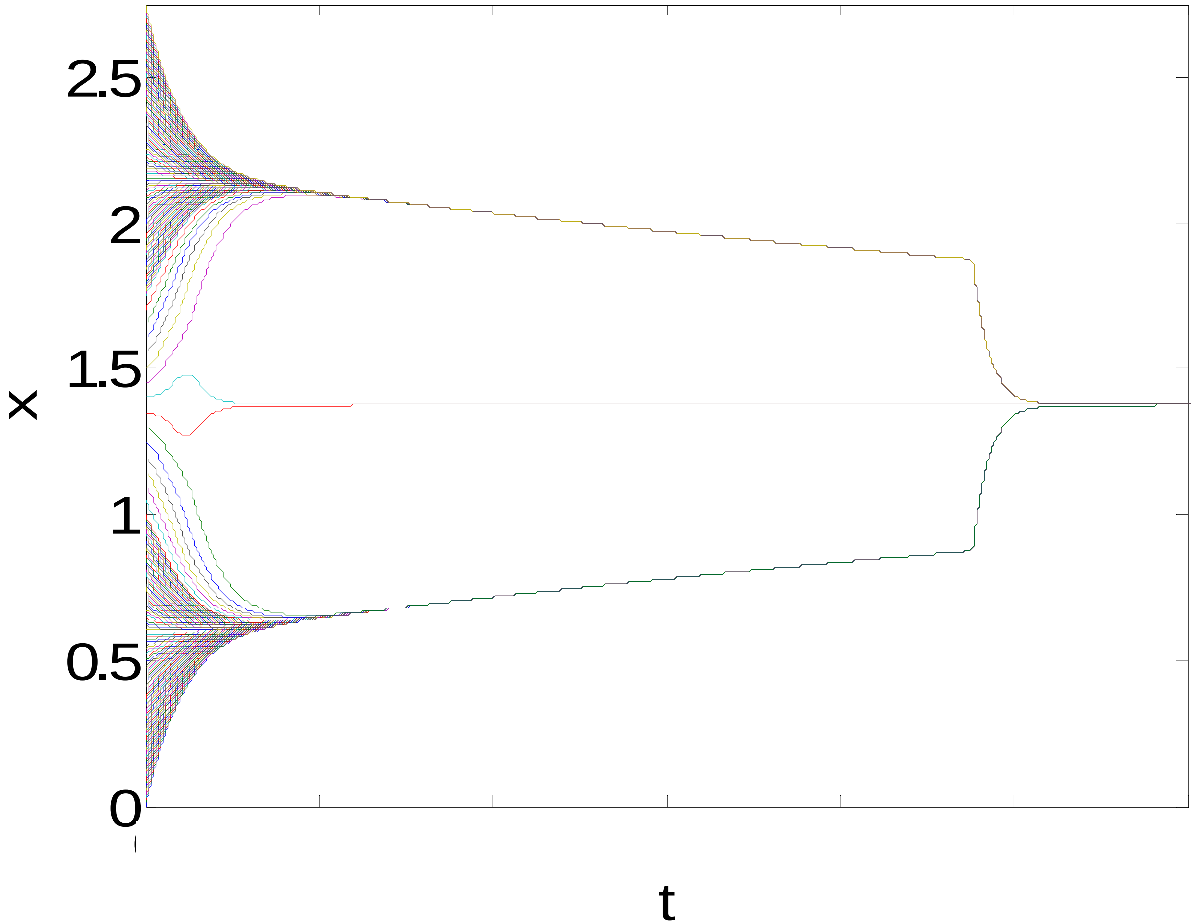


Convergence to unstable equilibrium possible



Large number of agents leads to a stable equilibrium

Same distribution



Conjecture

For any given “smooth” agent distribution,

P(convergence to stable equilibrium) $\rightarrow$ 1 when n $\rightarrow \infty$

Supported by

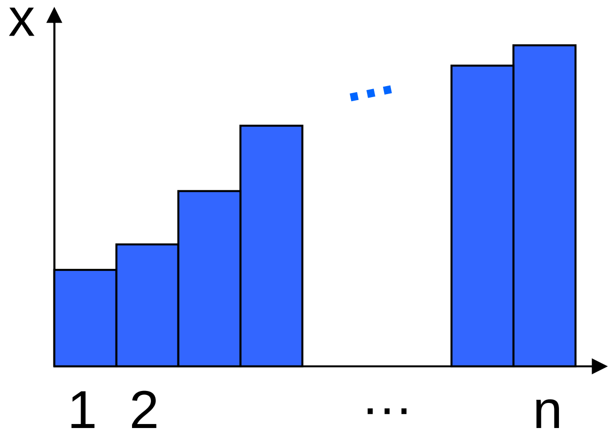
- Extensive numerical simulations
- Results for continuous time for ***continuum of agents***

Continuum of agents

Discrete agents

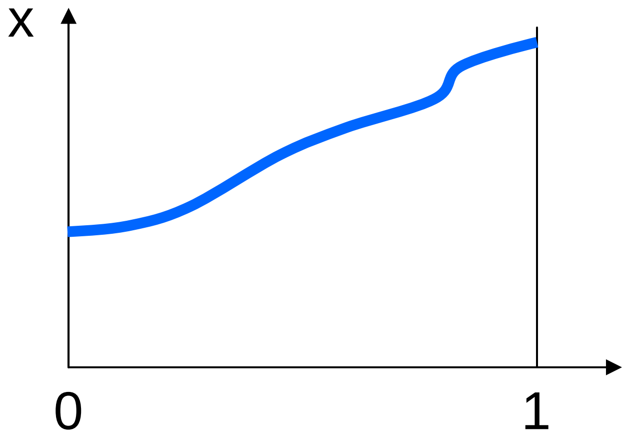
Opinions

$x \in \Re^n$ or $x : \{1, \ldots, n\} \to \Re$



Continuum of agents

Opinions $x : [0, 1] \to \Re$



Discrete agents:

$$\frac{d}{dt}x_i(t) = \sum_{j:|x_i(t)-x_j(t)|<1} (x_j(t) - x_i(t))$$

Continuum of agents:

$$\frac{d}{dt}x_t(\alpha) = \int_{\beta:|x_t(\alpha)-x_t(\beta)|<1} (x_t(\beta) - x_t(\alpha))$$

$$\frac{d}{dt}x_t(\alpha) = \int_{\beta:|x_t(\alpha)-x_t(\beta)|<1} (x_t(\beta) - x_t(\alpha))$$

Theorem: If the initial condition is regular, then there exists a unique solution, x_t is regular for all t and

$$\lim_{t\to\infty} x_t(\alpha) = s(\alpha)$$

with s piecewise constant

Theorem: For any two clusters of $s = \lim_{t\to\infty} x_t$, the inter-cluster distance is at least equal to

$$1 + \frac{\min(W_A, W_B)}{\max(W_A, W_B)}$$

Summary

- Simple (simplest?) multi-agent system with state-dependent communication topology.
- Proof of convergence and non-trivial clusters separation distance for a continuum of agents.
- Conjecture of a similar property for discrete agents.

Open problems

- All simulations show that a cluster separation of 2.2 arises from the dynamics.
- Hegselmann conjecture: only one cluster if sufficiently many agents.
- 2D case unexplored.