On one generic choice of transverse coordinates for a trajectory of a controlled mechanical system subject to non-holonomic constraints

Anton Shiriaev[†,⋆]   Leonid Freidovich[⋆]

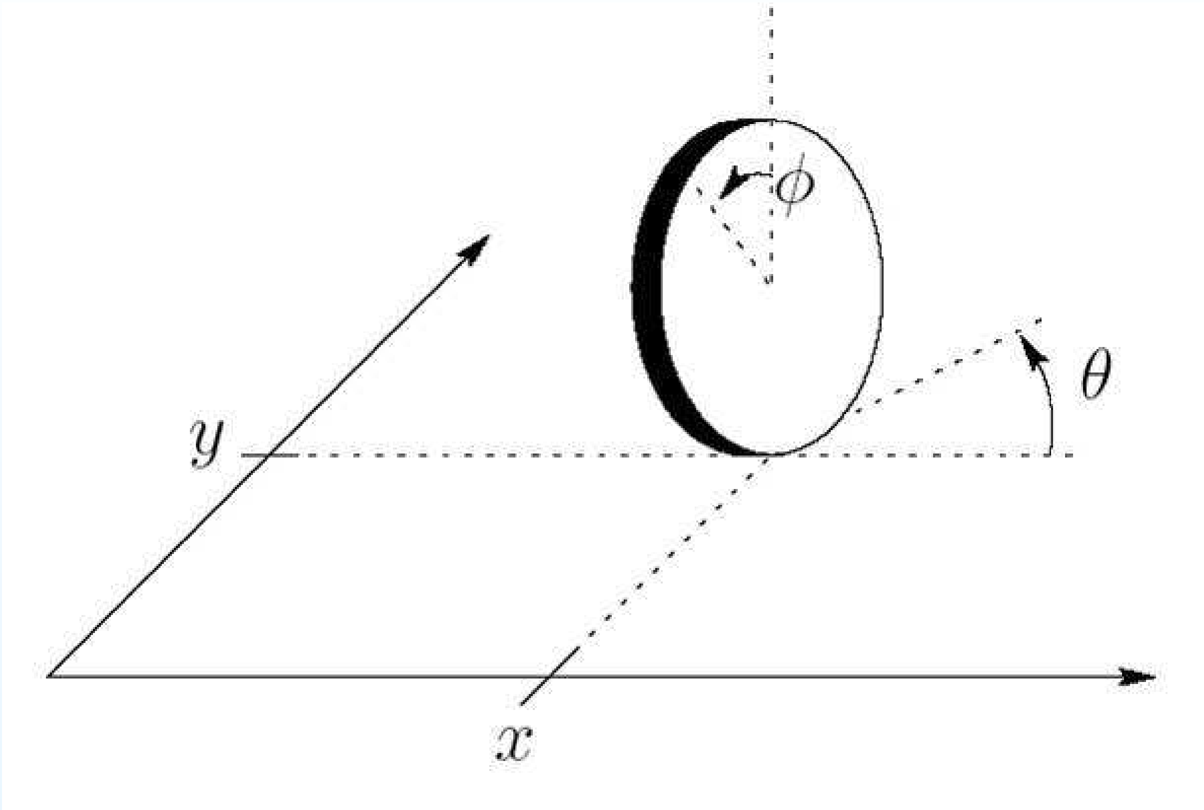
[⋆]Department of Applied Physics and Electronics
Umeå University, Sweden

[†]Department of Engineering Cybernetics
Norwegian University of Science and Technology
Trondheim, Norway

On one generic choice of transverse coordinates for a trajectory of a controlled mechanical system subject to non-holonomic constraints

- Motivation and Preliminaries
- Representation of a Motion for a Mechanical System
- Good Coordinates around a Target Motion
- Example

Motivating Example: a Unicycle



The equations of motion are

$$m\ddot{x} = F_x^c, \qquad m\ddot{y} = F_y^c, \qquad J\ddot{\theta} = u$$

Here F_x^c, F_y^c are components of constraint force; u is control

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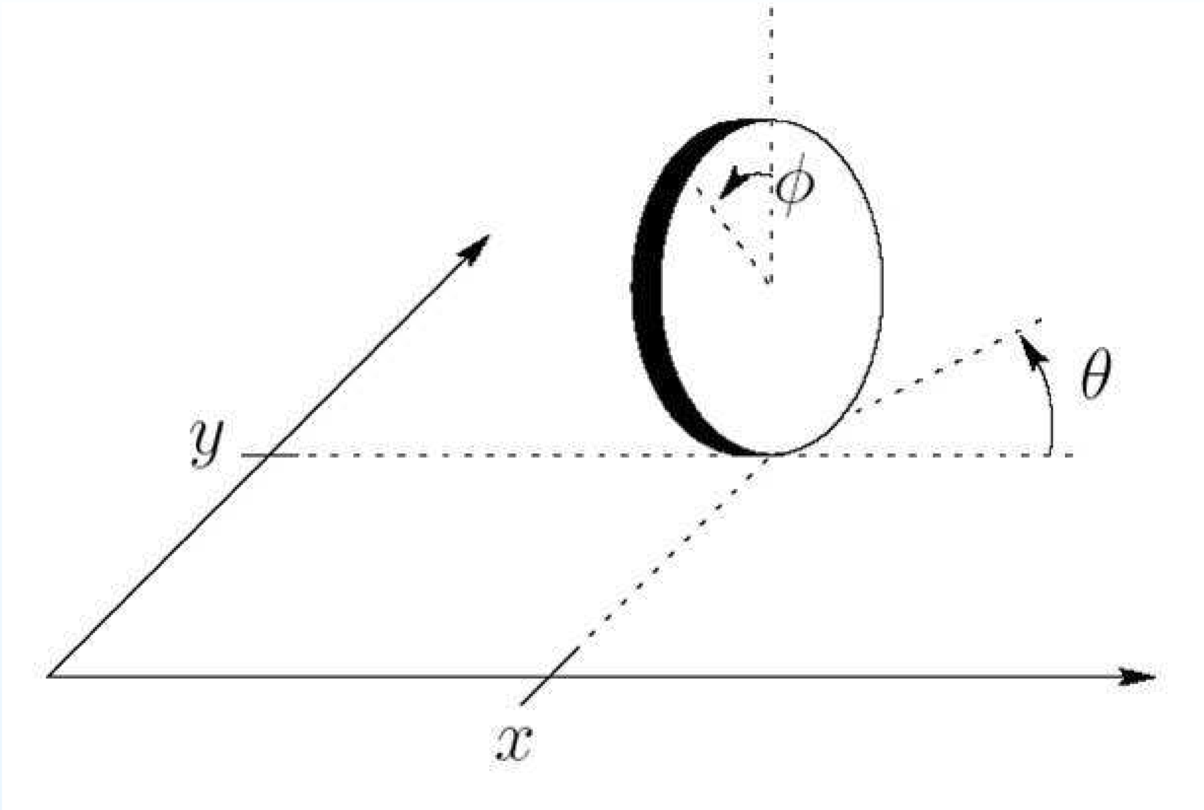
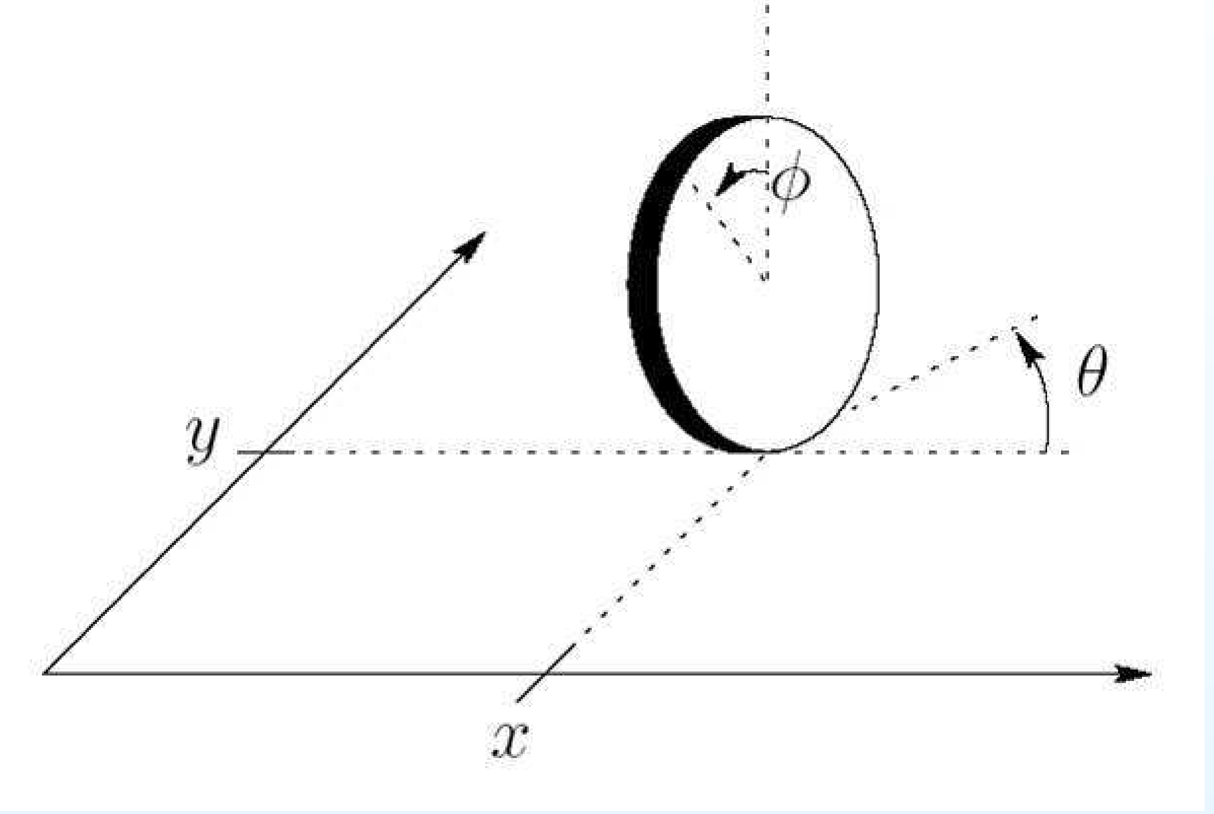


The equations of motion are

$$m\ddot{x} = \lambda \cdot \cos(\theta - \tfrac{\pi}{2}), \qquad m\ddot{y} = \lambda \cdot \sin(\theta - \tfrac{\pi}{2}), \qquad J\ddot{\theta} = u$$

Here λ is amplitude of the constraint force.

Motivating Example: a Unicycle



The equations of motion are

$$\begin{aligned}\ddot{x} &= -[\dot{y}\cdot\sin\theta + \dot{x}\cdot\cos\theta]\cdot\dot{\theta}\cdot\sin(\theta)\\ \ddot{y} &= [\dot{y}\cdot\sin\theta + \dot{x}\cdot\cos\theta]\cdot\dot{\theta}\cdot\cos(\theta)\\ J\ddot{\theta} &= u\end{aligned}$$

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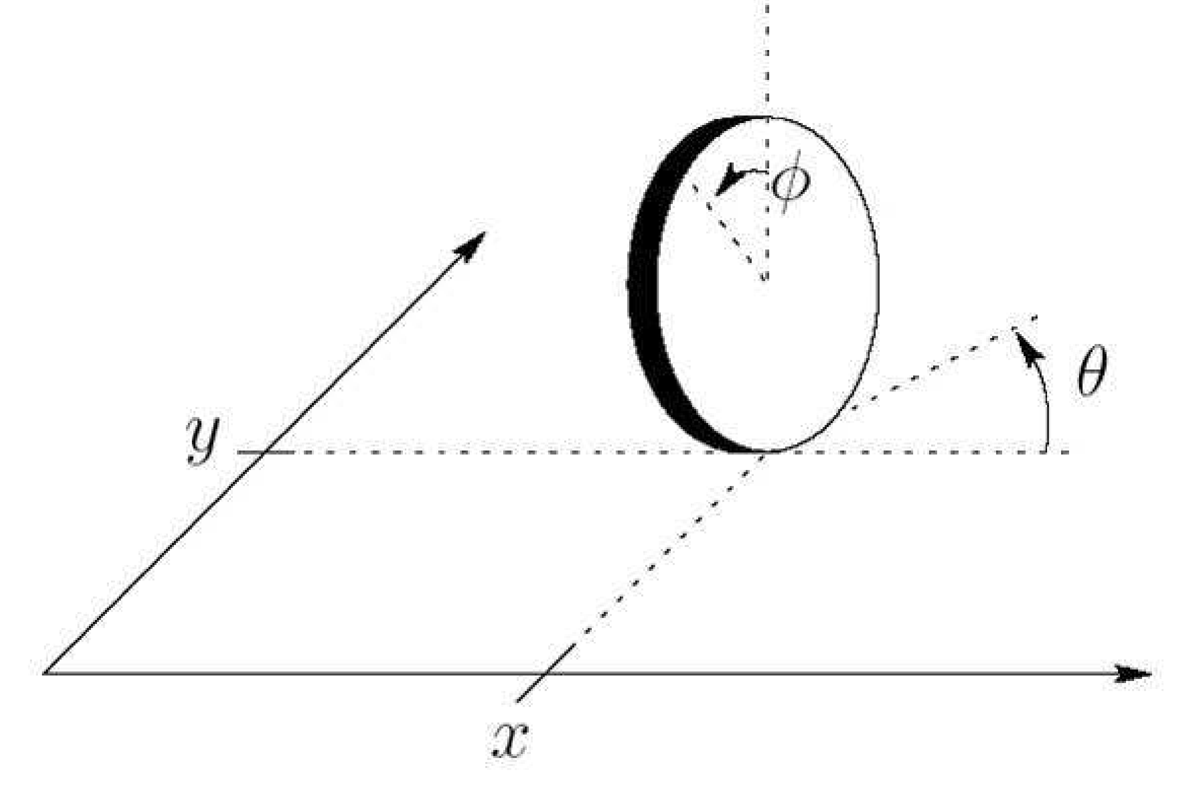
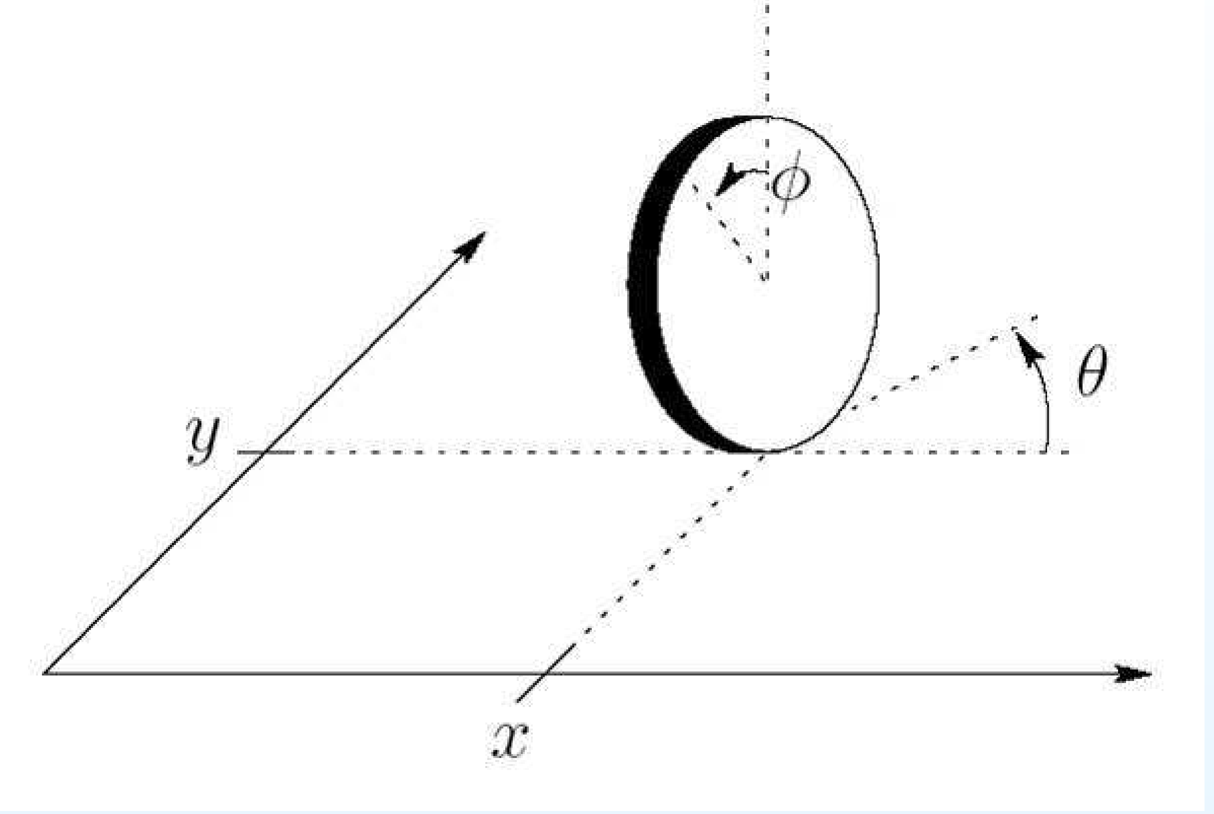


The equations of motion are

$$\frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}}\mathcal{L}\right] - \frac{\partial}{\partial q}\mathcal{L} = R(q,\dot{q}) + B(q)u, \qquad R_i = \dot{q}^T r_i(q)\dot{q}$$

Here $q \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $R(\cdot)$ is a vector of reaction forces.

Motivating Example: a Unicycle



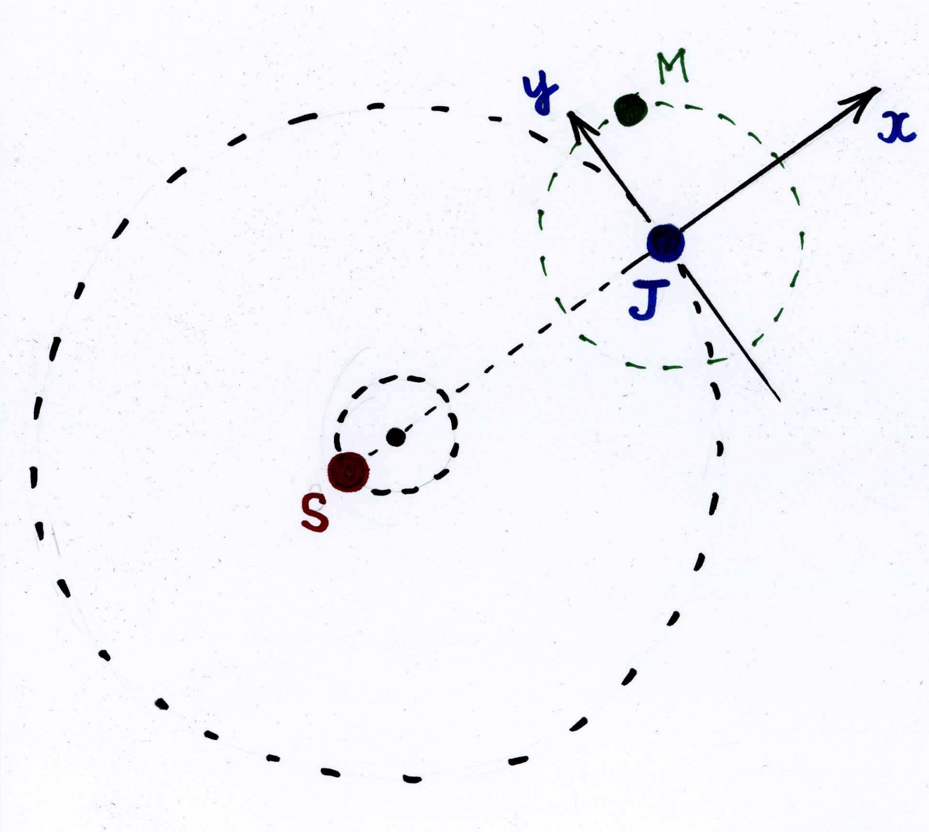
Problems:

- Given a motion, design controller for its orbital stabilization

- Given a motion and controller, analyze the dynamics

- Given specifications, plan a feasible motion

Motivating Example: Elements of Theory of G.W. Hill



Equations of motion for the position of the Moon in rotating coordinate frame are

$$\begin{cases} \ddot{x} - 2m\dot{y} &= \frac{\partial}{\partial x} F \\ \ddot{y} + 2m\dot{x} &= \frac{\partial}{\partial y} F \end{cases}$$

Here

$$F = \frac{\kappa}{\sqrt{x^2 + y^2}} + \frac{3}{2} m^2 x^2$$

m, κ are positive constants.

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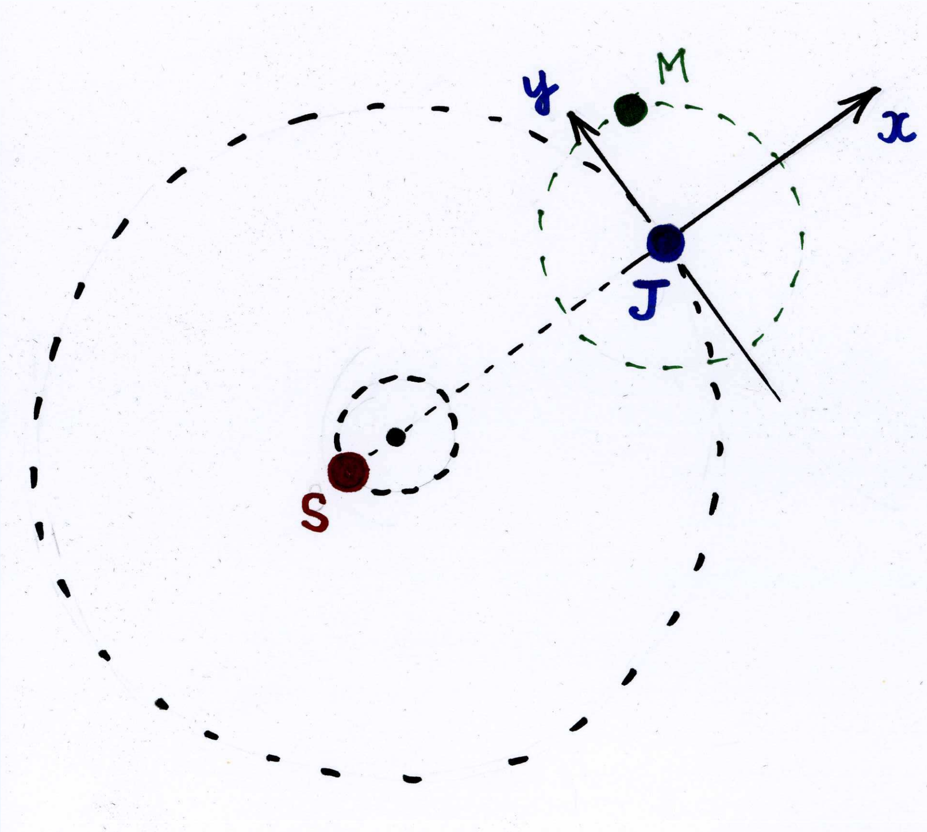

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The system has the invariant: $I = \dot{x}^2 + \dot{y}^2 - 2F(x, y) + C$

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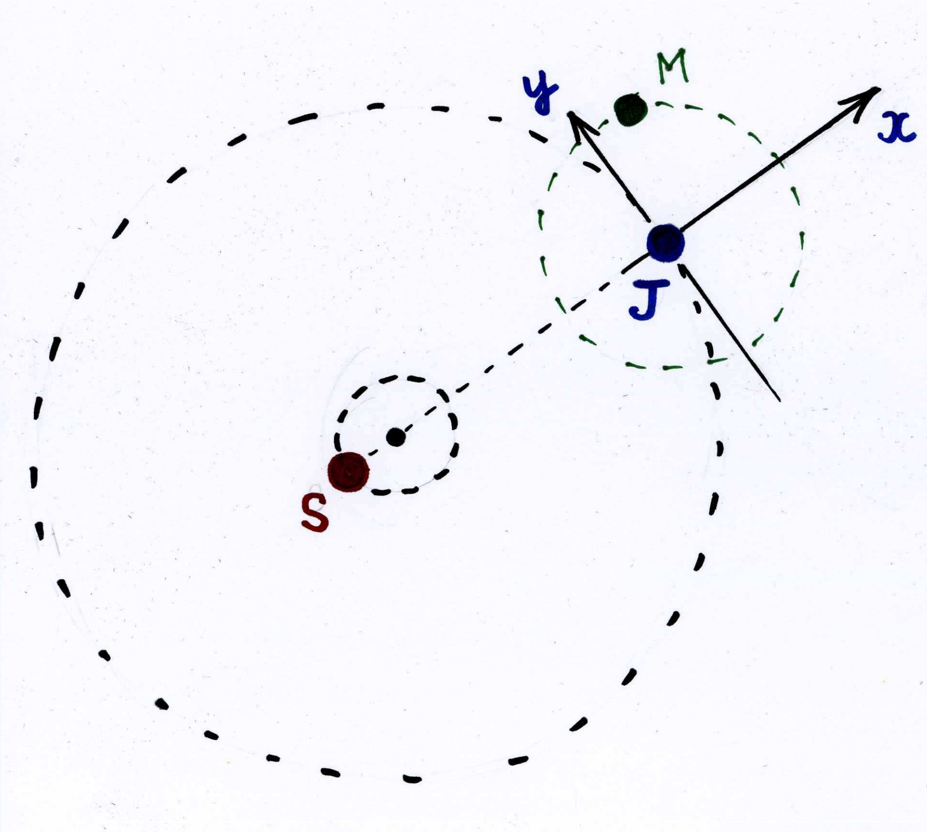

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Task: Analyze the dynamics in a vicinity of periodic circle motion

Motivating Example: Elements of Theory of G.W. Hill

Denote $[x_p(t),\ y_p(t)]$ the periodic solution

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Perturbed solutions $[x_p(t) + \delta x(t),\ y_p(t) + \delta y(t)]$ defined by

$$\frac{d^2}{dt^2}\left[\delta x\right] - 2m\frac{d}{dt}\left[\delta y\right] = \\ = \left[\frac{\partial^2}{\partial x^2}F(x_p(t), y_p(t))\right]\delta x + \left[\frac{\partial^2}{\partial x \partial y}F(x_p(t), y_p(t))\right]\delta y$$

$$\frac{d^2}{dt^2}\left[\delta y\right] + 2m\frac{d}{dt}\left[\delta x\right] = \\ = \left[\frac{\partial^2}{\partial x \partial y}F(x_p(t), y_p(t))\right]\delta x + \left[\frac{\partial^2}{\partial y^2}F(x_p(t), y_p(t))\right]\delta y$$

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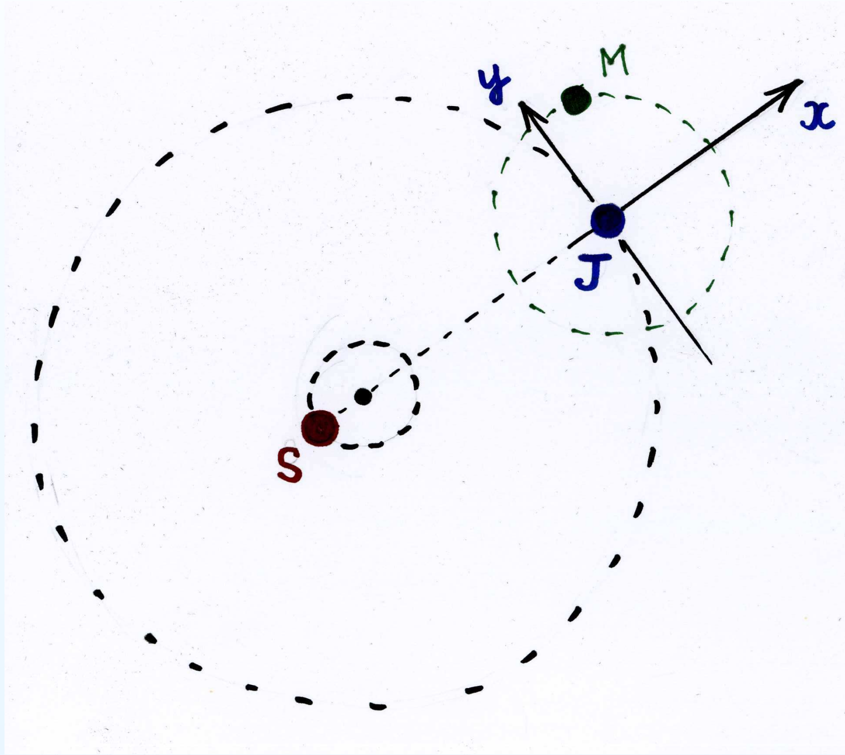
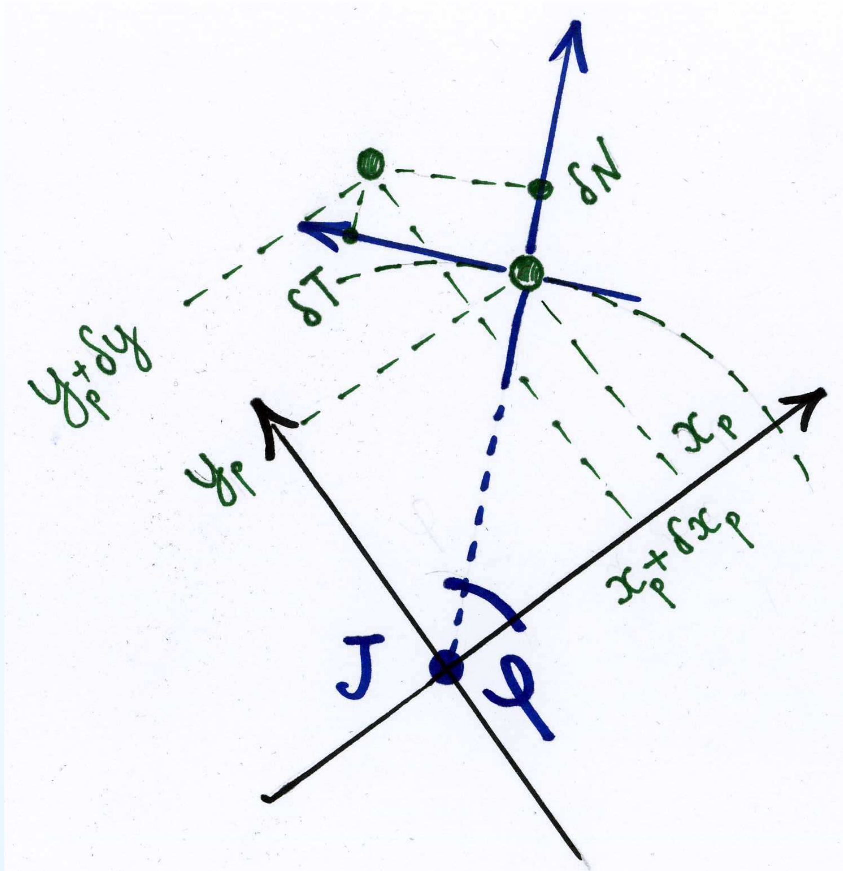
The integral Jacobi $I(\cdot)$ gives another relation

$$\frac{d}{dt}x_p(t)\frac{d}{dt}\left[\delta x\right] + \frac{d}{dt}y_p(t)\frac{d}{dt}\left[\delta y\right] =$$

$$= \left[\frac{\partial}{\partial x}F(x_p(t), y_p(t))\right]\delta x + \left[\frac{\partial}{\partial y}F(x_p(t), y_p(t))\right]\delta y$$

Motivating Example: Elements of Theory of G.W. Hill





Transform of coordinates into normal (δN) and tangent (δT)

$$\begin{bmatrix} \delta x \\ \delta y \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \delta T \\ \delta N \end{bmatrix}$$

Motivating Example: Elements of Theory of G.W. Hill

In a vicinity of the motion the original coordinates

$$\left[\boldsymbol{x},\ \boldsymbol{y},\ \dot{\boldsymbol{x}},\ \dot{\boldsymbol{y}}\right]$$

are changed into

$$\left[\boldsymbol{\phi},\ \boldsymbol{I},\ \boldsymbol{N},\ \dot{\boldsymbol{N}}\right]$$

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The linearization of $\boldsymbol{I}(\cdot)$ is straightforward: $\frac{d}{dt}\left[\delta \boldsymbol{I}\right] \equiv \boldsymbol{0}$

The linearization of $[N, \dot{N}]$ is the famous Hill's equation

$$\frac{d^2}{dt^2}\left[\delta N\right] + \Phi(t)\delta N = 0$$

Motivating Example: Observations

Analysis of dynamics in a vicinity of the motion's orbit requires:

- Decomposition of coordinates into
  - transverse to the trajectory $(\mathbf{dim} = \boldsymbol{2n - 1})$
  - along the trajectory $(\mathbf{dim} = \mathbf{1})$

  In the example they are

$$\left[\boldsymbol{I},\, \boldsymbol{N},\, \dot{\boldsymbol{N}}\right] \quad \text{and} \quad \boldsymbol{\phi}$$

Motivating Example: Observations

Analysis of dynamics in a vicinity of the motion's orbit requires:

- Decomposition of coordinates into
  - transverse to the trajectory $(\mathbf{dim} = \mathbf{2}\boldsymbol{n} - \mathbf{1})$
  - along the trajectory $(\mathbf{dim} = \mathbf{1})$

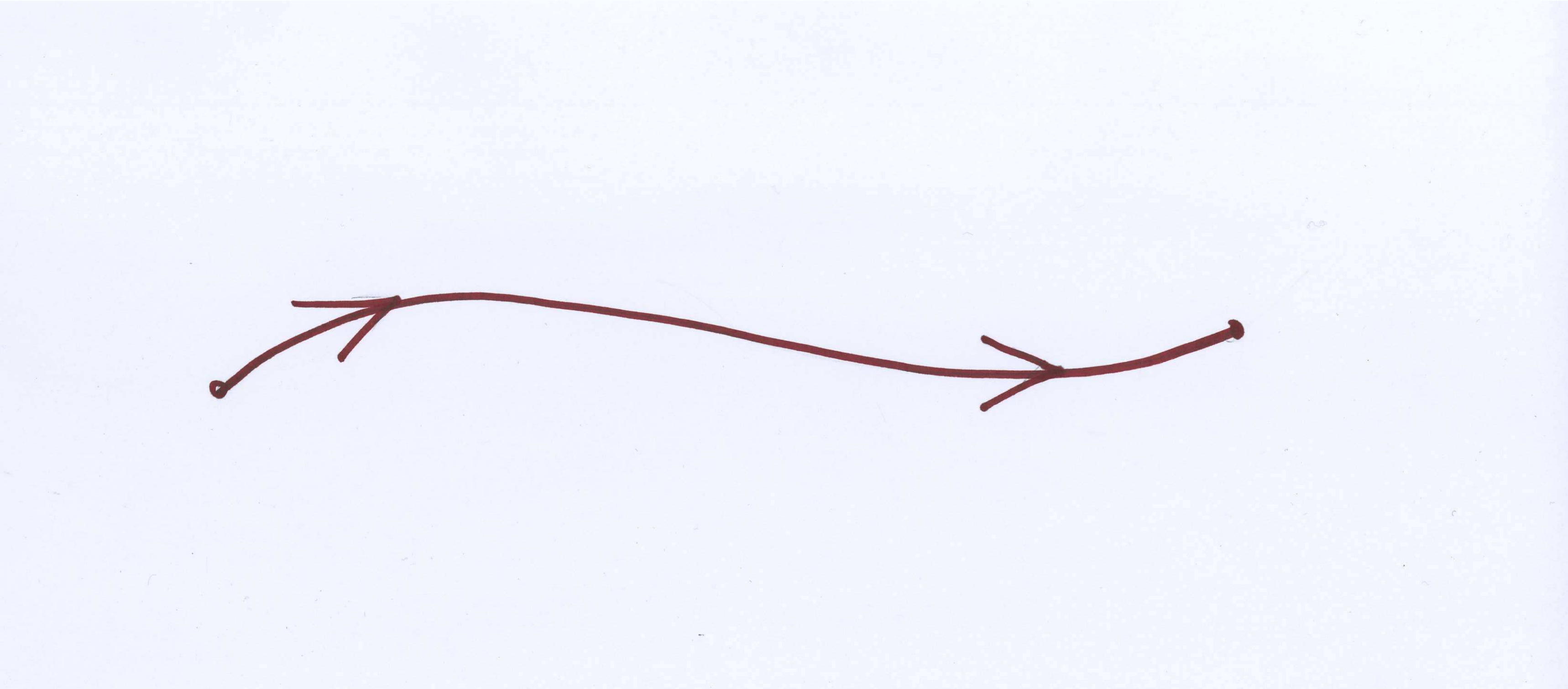
  In the example they are

$$\left[\boldsymbol{I},\ \boldsymbol{N},\ \dot{\boldsymbol{N}}\right] \quad \text{and} \quad \boldsymbol{\phi}$$

- Presence of invariants allows to reduce a number of transverse coordinates with non-trivial dynamics.

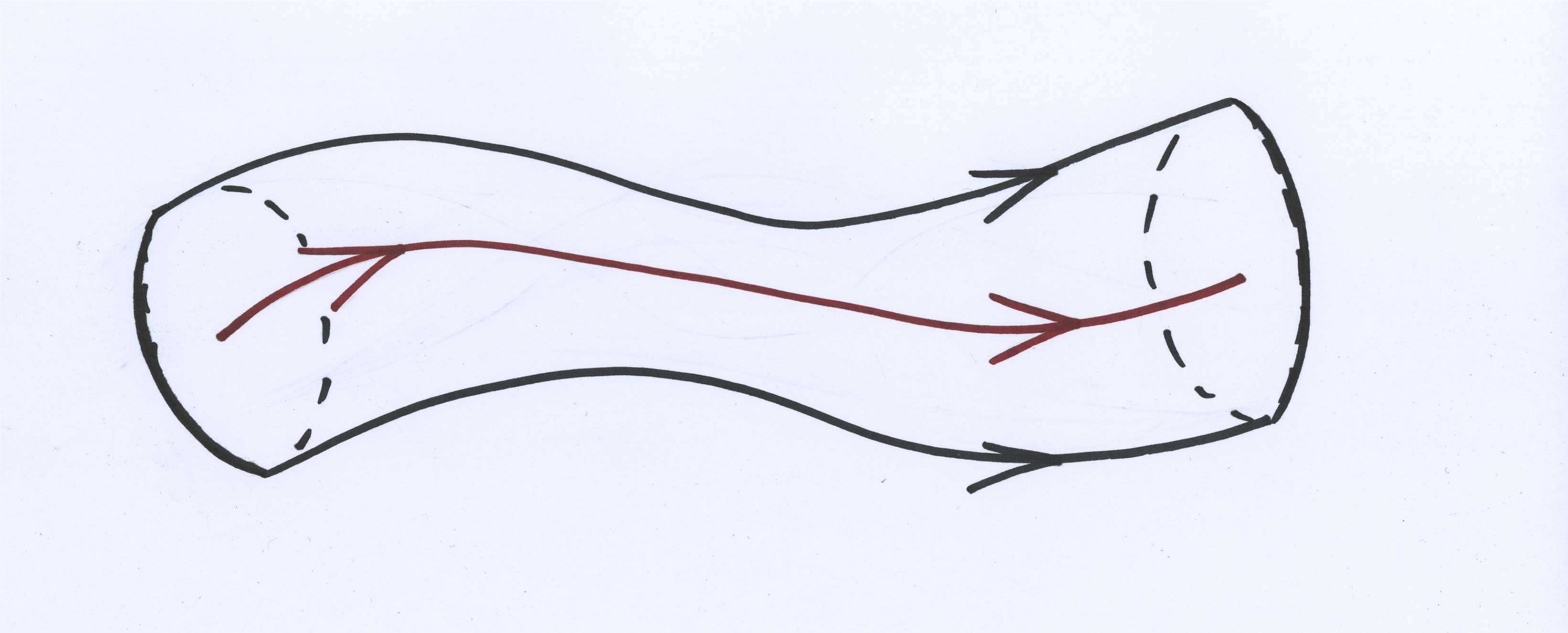
  In the example the integral Jacobi $\boldsymbol{I}(\cdot)$ is excluded.

Geometrical Interpretation



Given a trajectory of a nominal motion

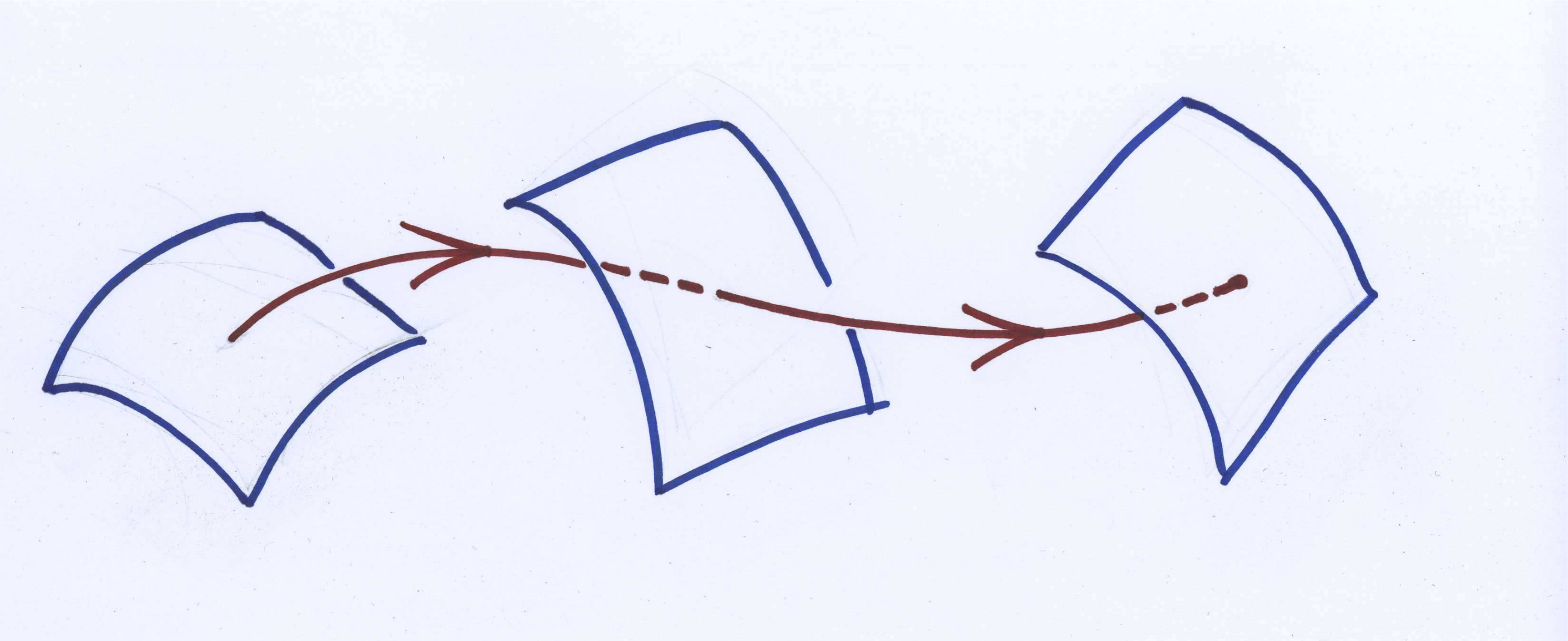
Geometrical Interpretation



We would like to analyze properties of the dynamics

in its tubing vicinity

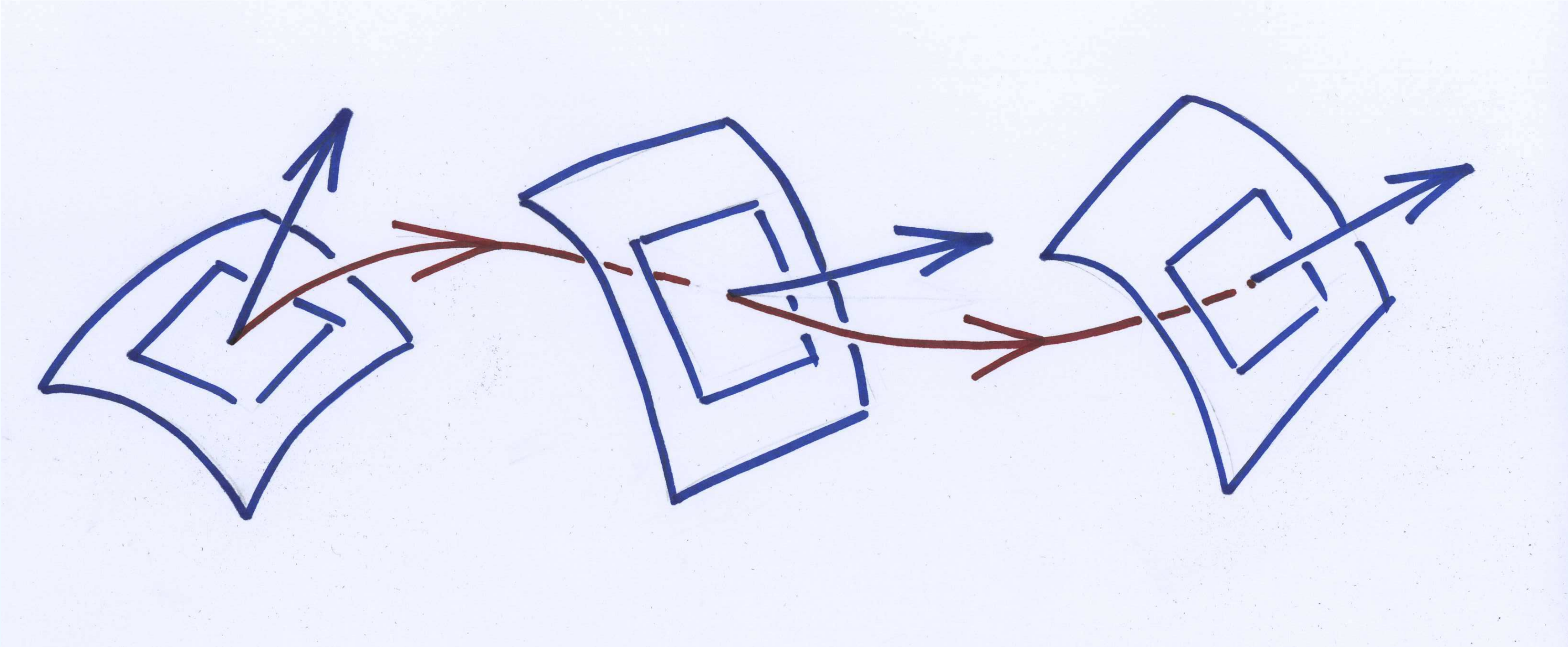
Geometrical Interpretation



Introduce a family of dis-joint transverse surfaces

that are continuously slicing this vicinity

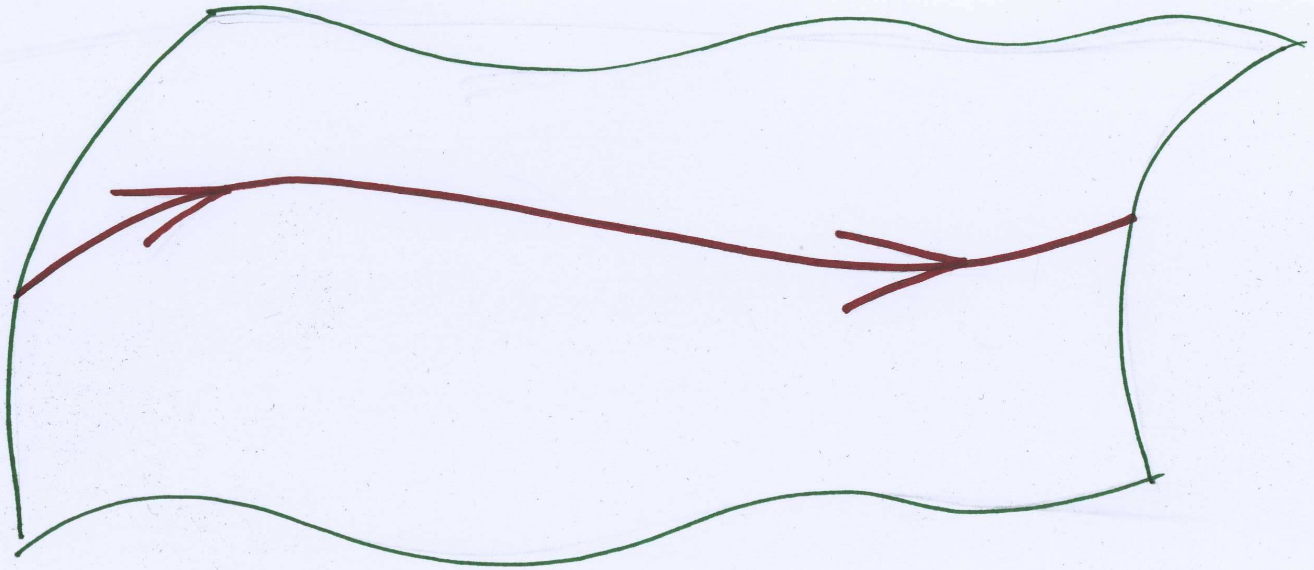
Geometrical Interpretation



For the linearization of the dynamics the surfaces

are substituted by tangent planes

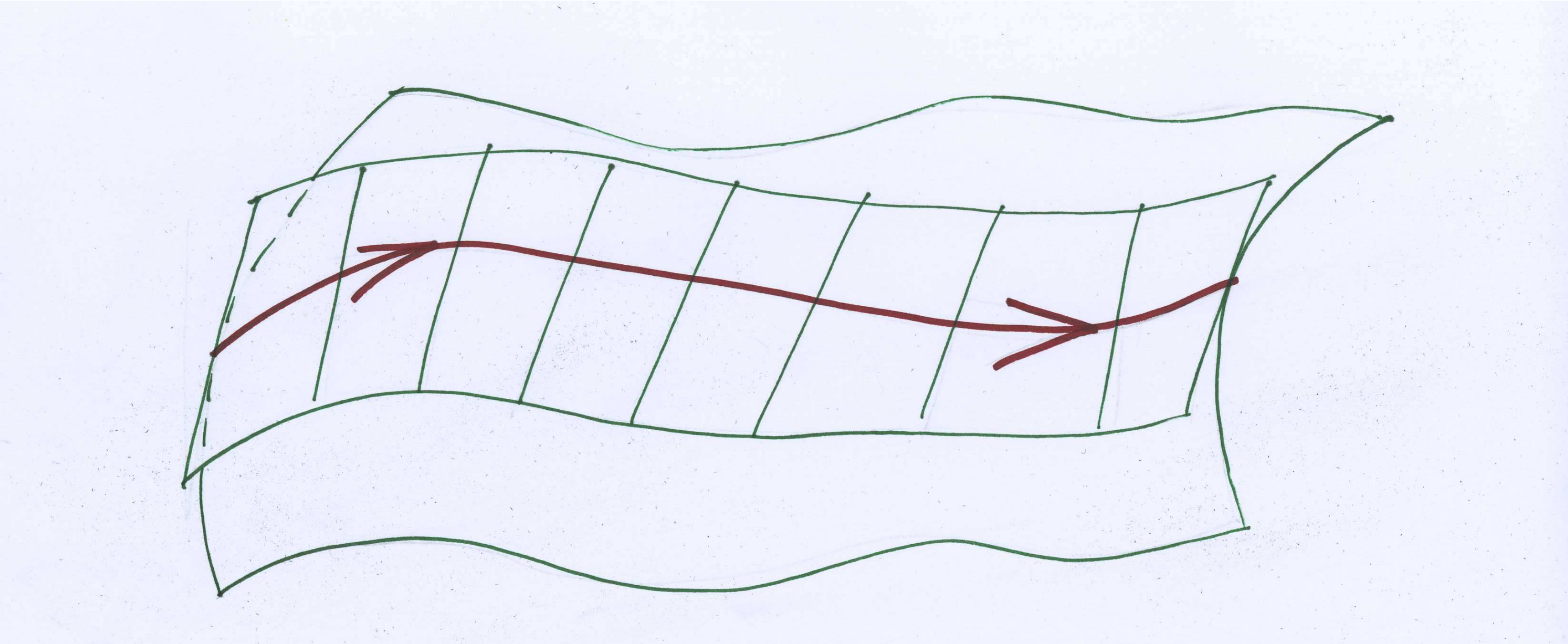
Geometrical Interpretation



If the dynamics have some invariants,

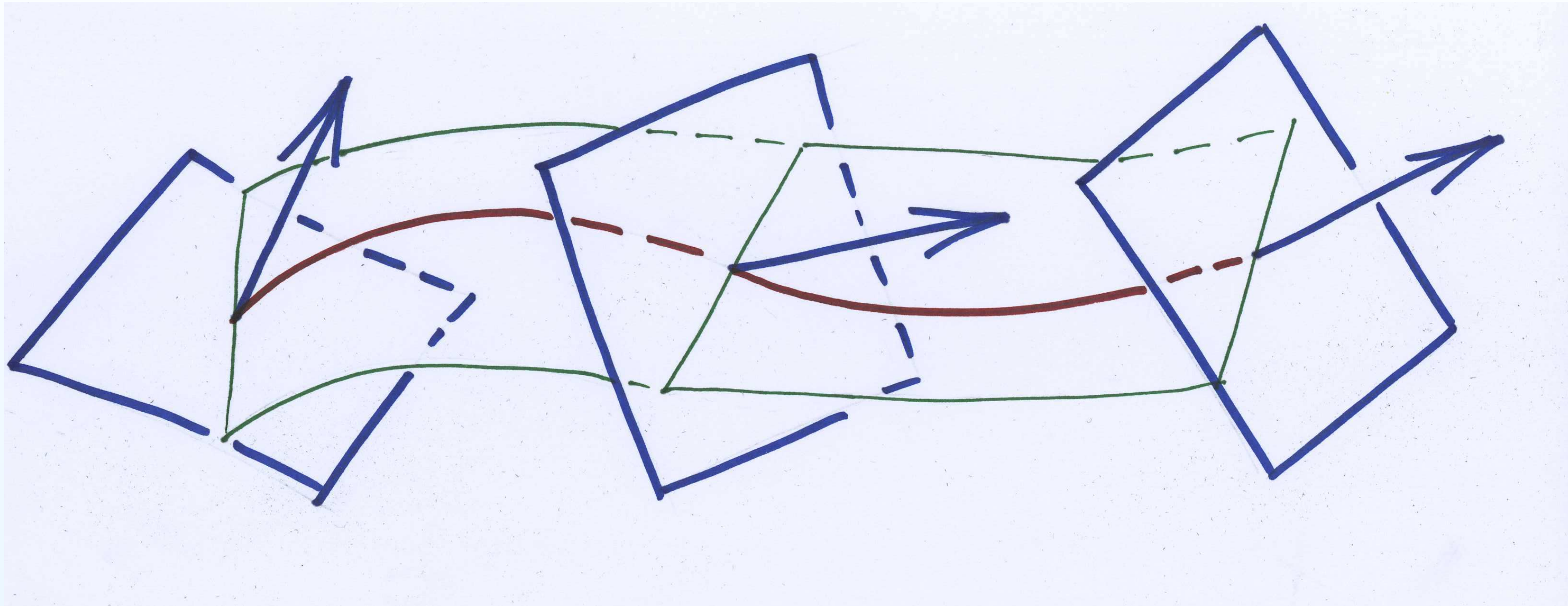
then they define a manifold

Geometrical Interpretation



For the linearization we consider the linear subspaces that are tangent to to the trajectory along this manifold

Geometrical Interpretation



Evolution of coordinates on these linear subspaces will define linearization of transverse coordinates with nontrivial behavior

Outline

- Motivation and Preliminaries
- Representation of a Motion for a Mechanical System
- Good Coordinates around a Target Motion
- Example

Representation for a Nominal Motion:

Given a model of mechanical system

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = R(q,\dot{q}) + B(q)u$$

where

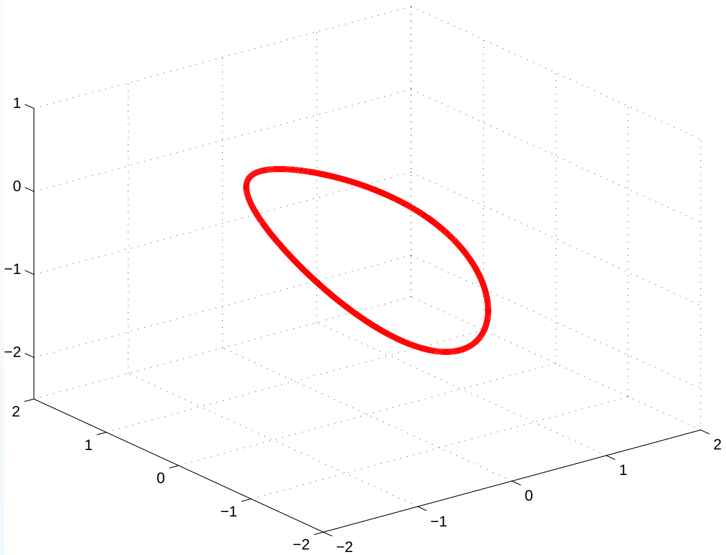
- $q = [q_1,\ q_2,\ \ldots,\ q_n]^T$ is a vector of degrees of freedom
- $u = [u_1,\ \ldots,\ u_m]^T$ is a vector of control forces
- $R(\cdot)$ is a vector

$$R(\cdot) = [R_1(\cdot),\ \ldots,\ R_n(\cdot)]^T$$

of reaction forces with

$$R_i(q,\dot{q}) = \dot{q}^T r_i(q)\dot{q}, \qquad i = 1,\ldots,n$$

Representation for a Nominal Motion (Cont'd):

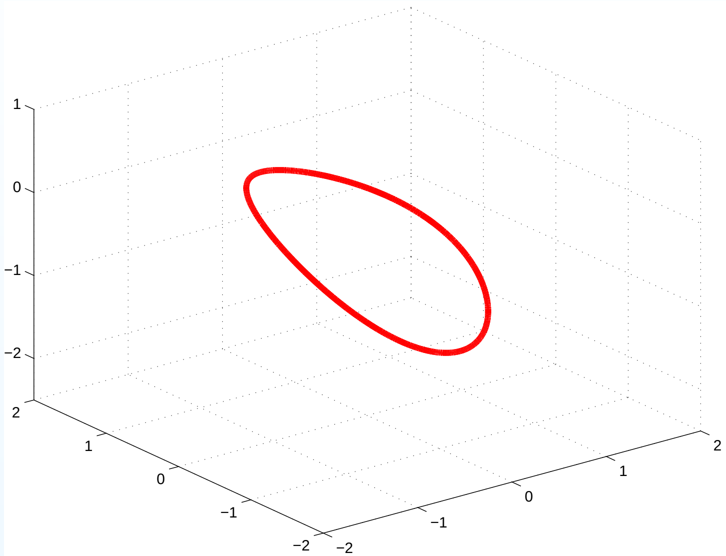


Given a motion

$$q^\star(t) = \Big(q_1^\star(t),\, q_2^\star(t),\, \ldots,\, q_n^\star(t)\Big)^T$$

of the system defined for the time interval $t \in [0, T]$

Representation for a Nominal Motion (Cont'd):



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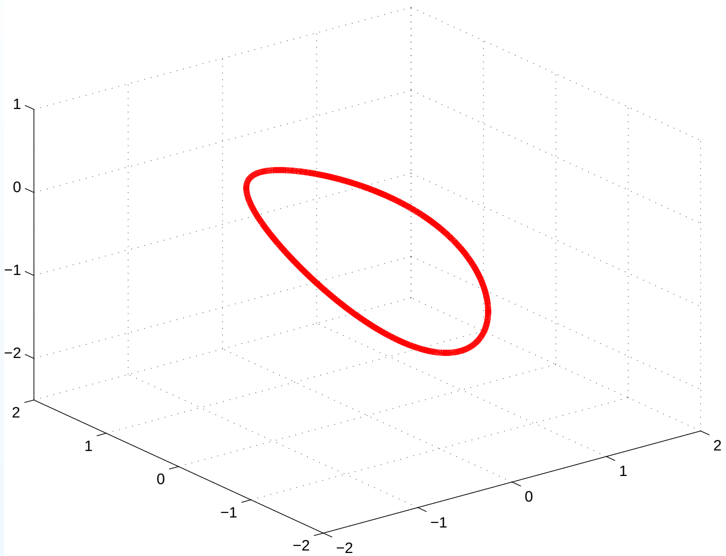
Then one can always find a way to re-parameterize the motion

- In the phase space the motion is the path

Representation for a Nominal Motion (Cont'd):



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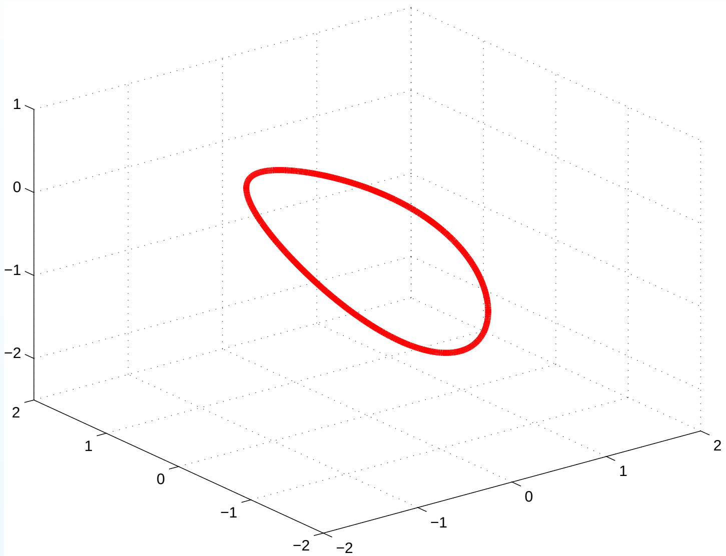
- In the phase space the motion is the path
- Denote by $\theta^\star(t)$ the arc-length along the path $\Rightarrow t = t(\theta^\star)$

Representation for a Nominal Motion (Cont'd):



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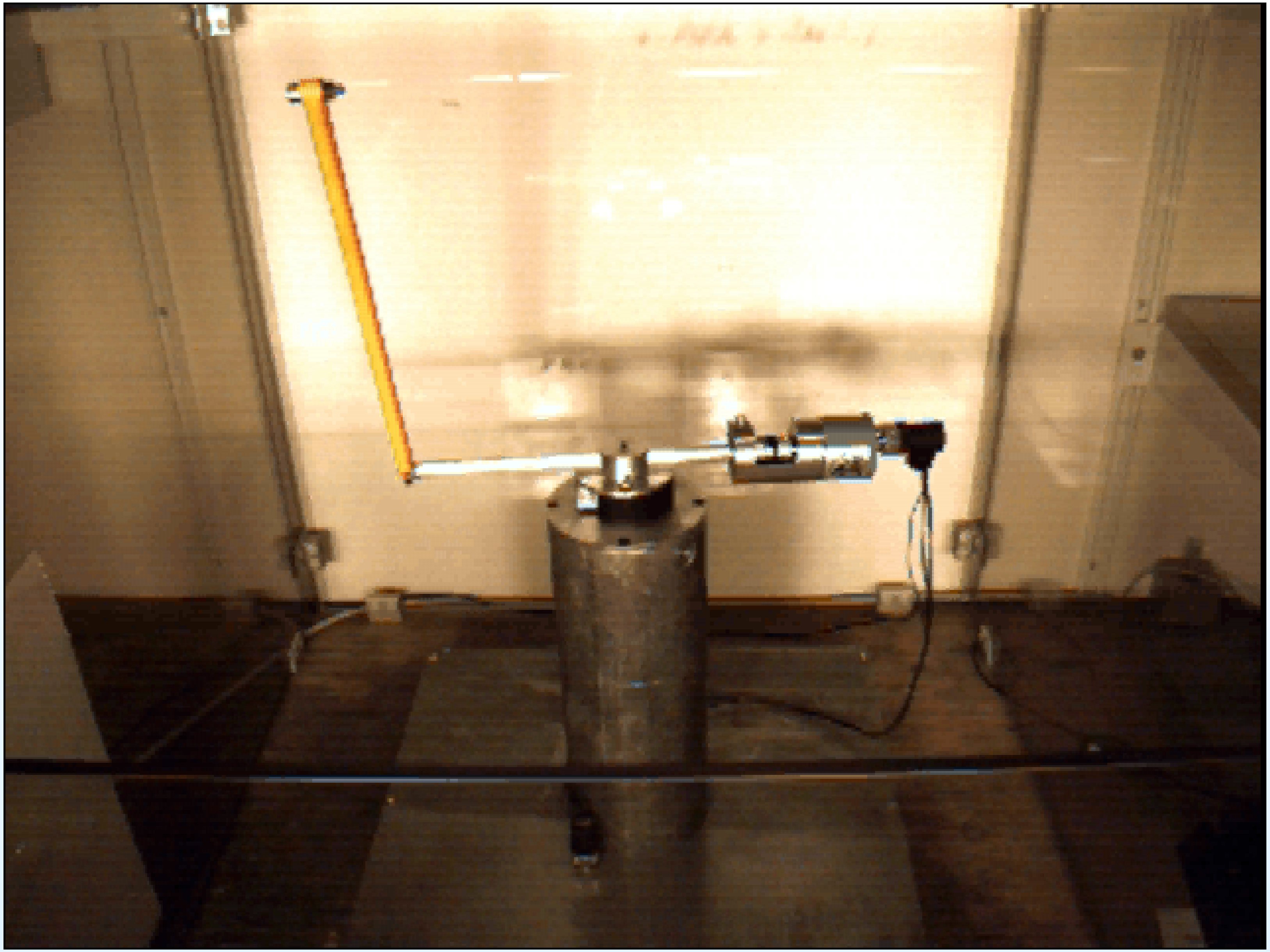
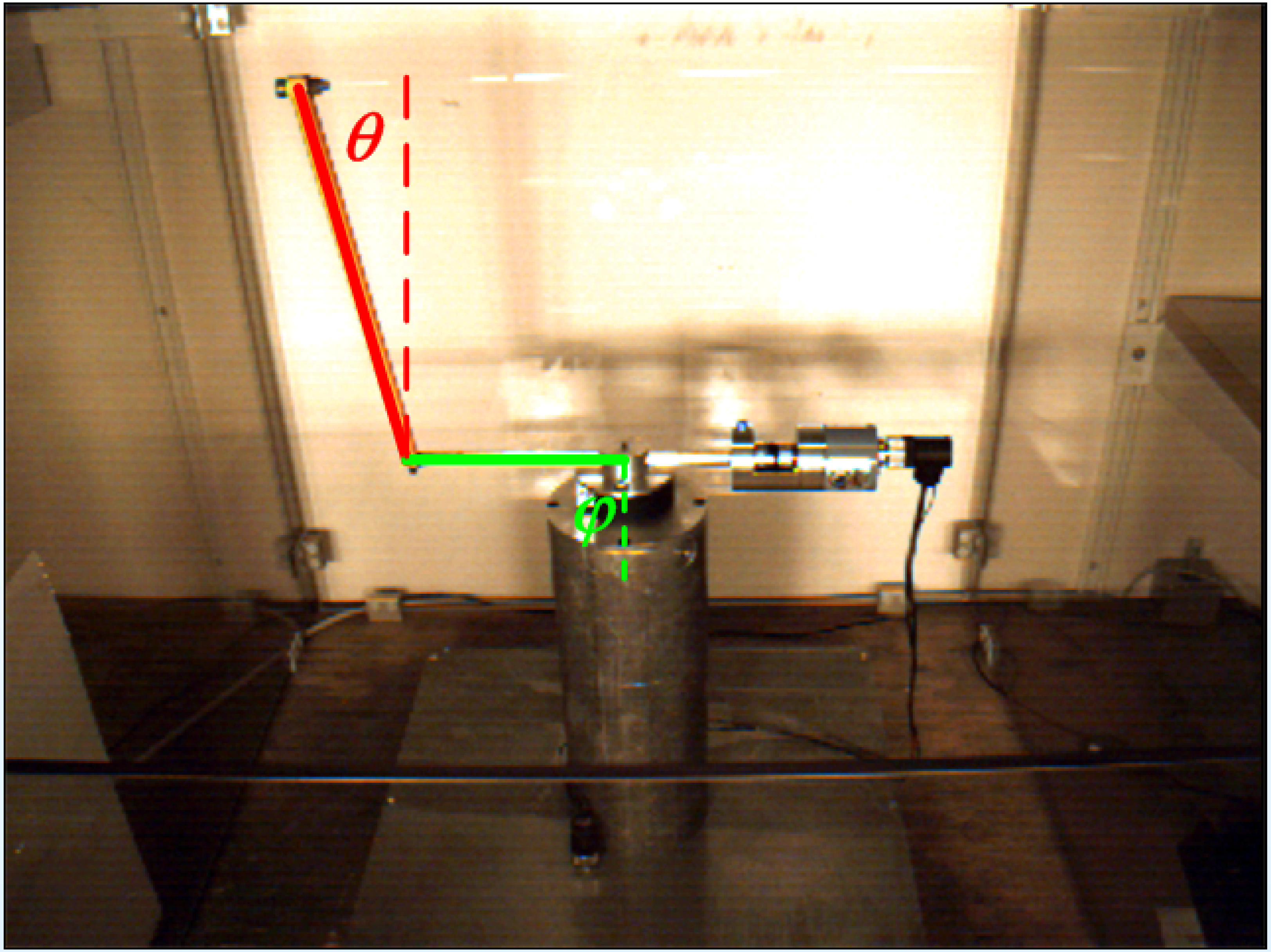
Then one can always find a way to re-parameterize the motion

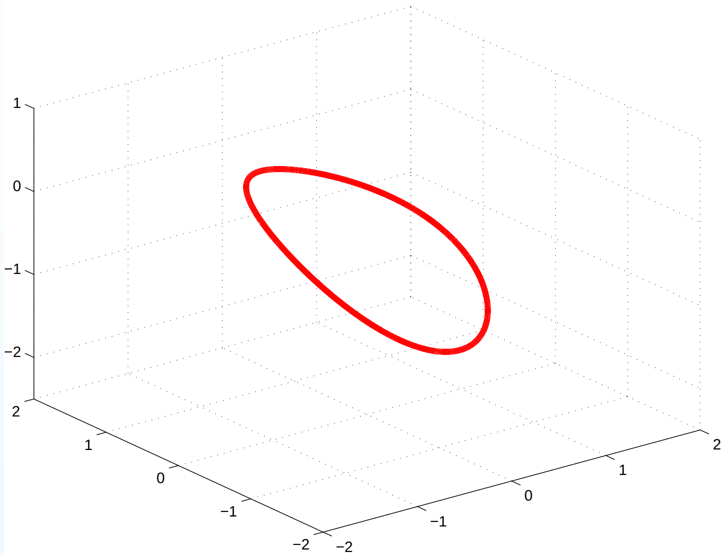
- In the phase space the motion is the path
- Denote by $\theta^\star(t)$ the arc-length along the path $\Rightarrow t = t(\theta^\star)$
- The motion is parameterized by this new variable $\theta^\star$

$$q_1^\star = q_1^\star(t(\theta^\star)) = \phi_1(\theta^\star), \quad \ldots, \quad q_n^\star = q_n^\star(t(\theta^\star)) = \phi_n(\theta^\star)$$




θ
φ

Representation for a Target Motion (Cont'd):



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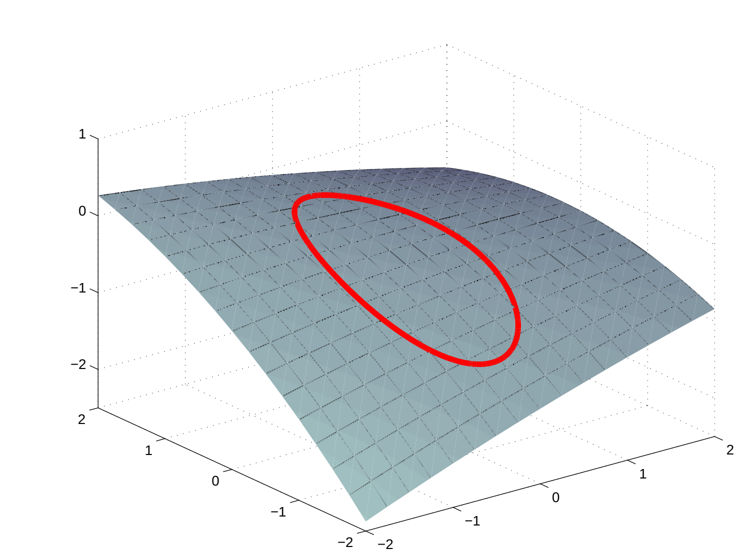
There are n-functions

$$\phi_1(\cdot),\ \phi_2(\cdot),\ \ldots,\ \phi_n(\cdot)$$

Representation for a Target Motion (Cont'd):



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The orbit of the motion lives on 2-dimensional manifold $[\theta, \dot{\theta}]$ defined by the relations

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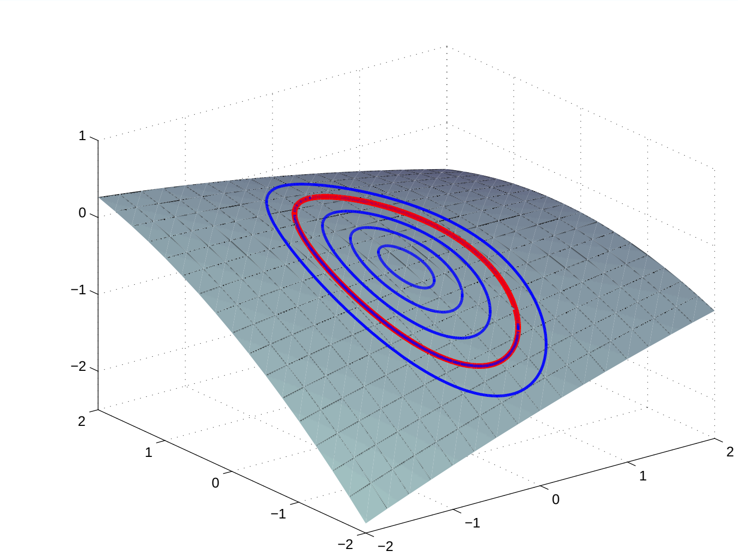

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How do the dynamics of θ look like on that manifold?

Reduced Dynamics: *Given an Euler-Lagrange system*

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\right] - \frac{\partial \mathcal{L}}{\partial q} = R(q, \dot{q}) + B(q)u$$

where the components of force $R(\cdot)$ are quadratic in $\dot{q}$ and

$$\dim q - \dim u = 1,$$

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where the components of force $R(\cdot)$ are quadratic in $\dot{q}$ and

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consider the following geometrical relations

$$q_1 = \phi_1(\theta), \quad q_2 = \phi_2(\theta), \quad \ldots, \quad q_n = \phi_n(\theta)$$

relating the coordinates q_i and the new independent variable θ.

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If there exists $u^*(\cdot)$ for the E-L system that makes these relations invariant along solutions of the closed loop system

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If there exists $u^*(\cdot)$ for the E-L system that makes these relations invariant along solutions of the closed loop system

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Then θ is a solution of the system

$$\alpha(\theta)\ddot{\theta} + \beta(\theta)\dot{\theta}^2 + \gamma(\theta) = 0$$

where $\alpha(\cdot)$, $\beta(\cdot)$ and $\gamma(\cdot)$ are scalar function. ■

Proof's Sketch: Invariance of the relations $q_i = \phi_i(\theta)$ implies

$$\begin{array}{rclcrcl} \dot q_1 & = & \phi_1'(\theta)\dot\theta, & \ldots, & \dot q_n & = & \phi_n'(\theta)\dot\theta \\ \ddot q_1 & = & \phi_1(\theta)''\dot\theta^2 + \phi_1'(\theta)\ddot\theta, & \ldots, & \ddot q_n & = & \phi_n(\theta)''\dot\theta^2 + \phi_n'(\theta)\ddot\theta \end{array}$$

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If the dynamics are

$$M(q)\begin{bmatrix} \ddot q_1 \\ \ddot q_2 \\ \vdots \\ \ddot q_n \end{bmatrix} + C(q,\dot q)\begin{bmatrix} \dot q_1 \\ \dot q_2 \\ \vdots \\ \dot q_n \end{bmatrix} + \begin{bmatrix} g_1(q) \\ g_2(q) \\ \vdots \\ g_n(q) \end{bmatrix} = \begin{bmatrix} \dot q^T r_1(q)\dot q \\ \dot q^T r_2(q)\dot q \\ \vdots \\ \dot q^T r_n(q)\dot q \end{bmatrix} + \begin{bmatrix} 0 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

Proof's Sketch: Invariance of the relations $q_i = \phi_i(\theta)$ implies

$$\begin{array}{lllllll} \dot q_1 &=& \phi_1'(\theta)\dot\theta, & \ldots, & \dot q_n &=& \phi_n'(\theta)\dot\theta \\ \ddot q_1 &=& \phi_1(\theta)''\dot\theta^2 + \phi_1'(\theta)\ddot\theta, & \ldots, & \ddot q_n &=& \phi_n(\theta)''\dot\theta^2 + \phi_n'(\theta)\ddot\theta \end{array}$$

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Then one picks up the first equation (independent on control)

$$m_{11}(q)\ddot q_1 + \cdots + m_{1n}(q)\ddot q_n + $$
$$+ c_{11}(q,\dot q)\dot q_1 + \cdots + c_{1n}(q,\dot q)\dot q_n + g_1(q) = \dot q^T r_1(q)\dot q + \mathbf{0}$$

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The matrix-function $C(q, \dot{q})$ of the E-L system

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = R(q, \dot{q}) + B(q)u^*,$$

is linear in $\dot{q}$.

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Substituting expressions for q, $\dot q$, and $\ddot q$ into the system dynamics, we obtain the system of n second order equations

$$M(\Phi)\left[\Phi'\ddot\theta + \Phi''\dot\theta^2\right] + C(\Phi, \Phi'\dot\theta)\Phi'\dot\theta + G(\Phi) = R(\Phi, \Phi'\dot\theta) + B(\Phi)u^*$$

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Substituting expressions for q, $\dot q$, and $\ddot q$ into the system dynamics, we obtain the system of n second order equations

$$M(\Phi)\left[\Phi'\ddot\theta + \Phi''\dot\theta^2\right] + C(\Phi,\Phi'\dot\theta)\Phi'\dot\theta + G(\Phi) = R(\Phi,\Phi'\dot\theta) + B(\Phi)u^*$$

where $\Phi(\theta)$, $\Phi'(\theta)$ and $\Phi''(\theta)$ denote the vectors

$$\begin{aligned}\Phi(\theta) &= \left[\phi_1(\theta),\ \phi_2(\theta),\ \ldots,\phi_n(\theta)\right]^T\\ \Phi'(\theta) &= \left[\phi_1'(\theta),\ \phi_2'(\theta),\ \ldots,\phi_n'(\theta)\right]^T\\ \Phi''(\theta) &= \left[\phi_1''(\theta),\ \phi_2''(\theta),\ \ldots,\phi_n''(\theta)\right]^T\end{aligned}$$

Proof's Sketch (cont'd): Since $\operatorname{rank} B(q) = n - 1$, then there exists a $1 \times n$ raw function $B^{\perp}(q)$ such that

$$B^{\perp}(q) B(q) u^{*} = 0, \quad \forall\, q$$

Proof's Sketch (cont'd): Since rank $B(q) = n - 1$, then there exists a $1 \times n$ raw function $B^{\perp}(q)$ such that

$$B^{\perp}(q)B(q)u^{*} = 0, \quad \forall\, q$$

Then the functions $\alpha(\cdot)$, $\beta(\cdot)$ and $\gamma(\cdot)$ of the system

$$\alpha(\theta)\ddot{\theta} + \beta(\theta)\dot{\theta}^{2} + \gamma(\theta) = 0$$

can be computed as follows

$$\begin{aligned}
\alpha(\theta) &= B^{\perp}(\Phi(\theta))\, M(\Phi(\theta))\, \Phi'(\theta) \\
\beta(\theta) &= B^{\perp}(\Phi(\theta)) \big[M(\Phi(\theta))\, \Phi''(\theta) + \\
&\qquad + C(\Phi(\theta), \Phi'(\theta))\Phi'(\theta) - R(\Phi(\theta), \Phi'(\theta)) \big] \\
\gamma(\theta) &= B^{\perp}(\Phi(\theta))\, G\,(\Phi(\theta))
\end{aligned}$$

Comments on Underactuation $= \mathbf{2}$:

Invariance of the relations $q_i = \phi_i(\theta)$ implies

$$\begin{aligned} \dot q_1 &= \phi_1'(\theta)\dot\theta, & \ldots, \quad \dot q_n &= \phi_n'(\theta)\dot\theta \\ \ddot q_1 &= \phi_1(\theta)''\dot\theta^2 + \phi_1'(\theta)\ddot\theta, & \ldots, \quad \ddot q_n &= \phi_n(\theta)''\dot\theta^2 + \phi_n'(\theta)\ddot\theta \end{aligned}$$

Comments on Underactuation $= 2$:

Invariance of the relations $q_i = \phi_i(\theta)$ implies

$$\begin{aligned} \dot q_1 &= \phi_1'(\theta)\dot\theta, & \ldots, \quad \dot q_n &= \phi_n'(\theta)\dot\theta \\ \ddot q_1 &= \phi_1(\theta)''\dot\theta^2 + \phi_1'(\theta)\ddot\theta, & \ldots, \quad \ddot q_n &= \phi_n(\theta)''\dot\theta^2 + \phi_n'(\theta)\ddot\theta \end{aligned}$$

If the dynamics are

$$M(q)\begin{bmatrix}\ddot q_1\\ \ddot q_2\\ \vdots\\ \ddot q_n\end{bmatrix} + C(q,\dot q)\begin{bmatrix}\dot q_1\\ \dot q_2\\ \vdots\\ \dot q_n\end{bmatrix} + \begin{bmatrix}g_1(q)\\ g_2(q)\\ \vdots\\ g_n(q)\end{bmatrix} = \begin{bmatrix}\dot q^T r_1(q)\dot q\\ \dot q^T r_2(q)\dot q\\ \vdots\\ \dot q^T r_n(q)\dot q\end{bmatrix} + \begin{bmatrix}0\\ 0\\ u_3\\ \vdots\end{bmatrix}$$

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Then one picks up two first equations (independent on control)

$$m_{11}(q)\ddot q_1 + \cdots + m_{1n}(q)\ddot q_n + \\ + c_{11}(q,\dot q)\dot q_1 + \cdots + c_{1n}(q,\dot q)\dot q_n + g_1(q) = \dot q^T r_1(q)\dot q + \mathbf{0}$$

$$m_{21}(q)\ddot q_1 + \cdots + m_{2n}(q)\ddot q_n + \\ + c_{21}(q,\dot q)\dot q_1 + \cdots + c_{2n}(q,\dot q)\dot q_n + g_2(q) = \dot q^T r_2(q)\dot q + \mathbf{0}$$

Comments on Underactuation $= \mathbf{2}$:

Invariance of the relations $q_i = \phi_i(\theta)$ implies

$$\begin{array}{lllllll} \dot{q}_1 & = & \phi_1'(\theta)\dot{\theta}, & \ldots, & \dot{q}_n & = & \phi_n'(\theta)\dot{\theta} \\ \ddot{q}_1 & = & \phi_1(\theta)''\dot{\theta}^2 + \phi_1'(\theta)\ddot{\theta}, & \ldots, & \ddot{q}_n & = & \phi_n(\theta)''\dot{\theta}^2 + \phi_n'(\theta)\ddot{\theta} \end{array}$$

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Then two first equations (independent on control) result in

$$\begin{array}{lll} \alpha_1(\theta)\ddot{\theta} + \beta_1(\theta)\dot{\theta}^2 + \gamma_1(\theta) & = & \mathbf{0} \\ \alpha_2(\theta)\ddot{\theta} + \beta_2(\theta)\dot{\theta}^2 + \gamma_2(\theta) & = & \mathbf{0} \end{array}$$

and $\theta(t)$ is the solution of both equations!

Integral of Motion: *Suppose the solution*

$$\theta(t) = \theta(t, \theta_0, \dot{\theta}_0)$$

of the system

$$\alpha(\theta)\ddot{\theta} + \beta(\theta)\dot{\theta}^2 + \gamma(\theta) = 0$$

with initial conditions $[\theta_0, \dot{\theta}_0]$ *and* $\alpha(\theta_0) \neq 0$ *exists.*

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with initial conditions $[\theta_0, \dot{\theta}_0]$ *and* $\alpha(\theta_0) \neq 0$ *exists.*

Then the function

$$I(\theta, \dot{\theta}, \theta_0, \dot{\theta}_0) = \dot{\theta}^2 - \psi(\theta_0, \theta)\left[\dot{\theta}_0^2 - \int_{\theta_0}^{\theta} \psi(s, \theta_0)\frac{2\gamma(s)}{\alpha(s)}ds\right]$$

with

$$\psi(\theta_0, \theta_1) = \exp\left\{-2\int_{\theta_0}^{\theta_1} \frac{\beta(\tau)}{\alpha(\tau)}d\tau\right\}$$

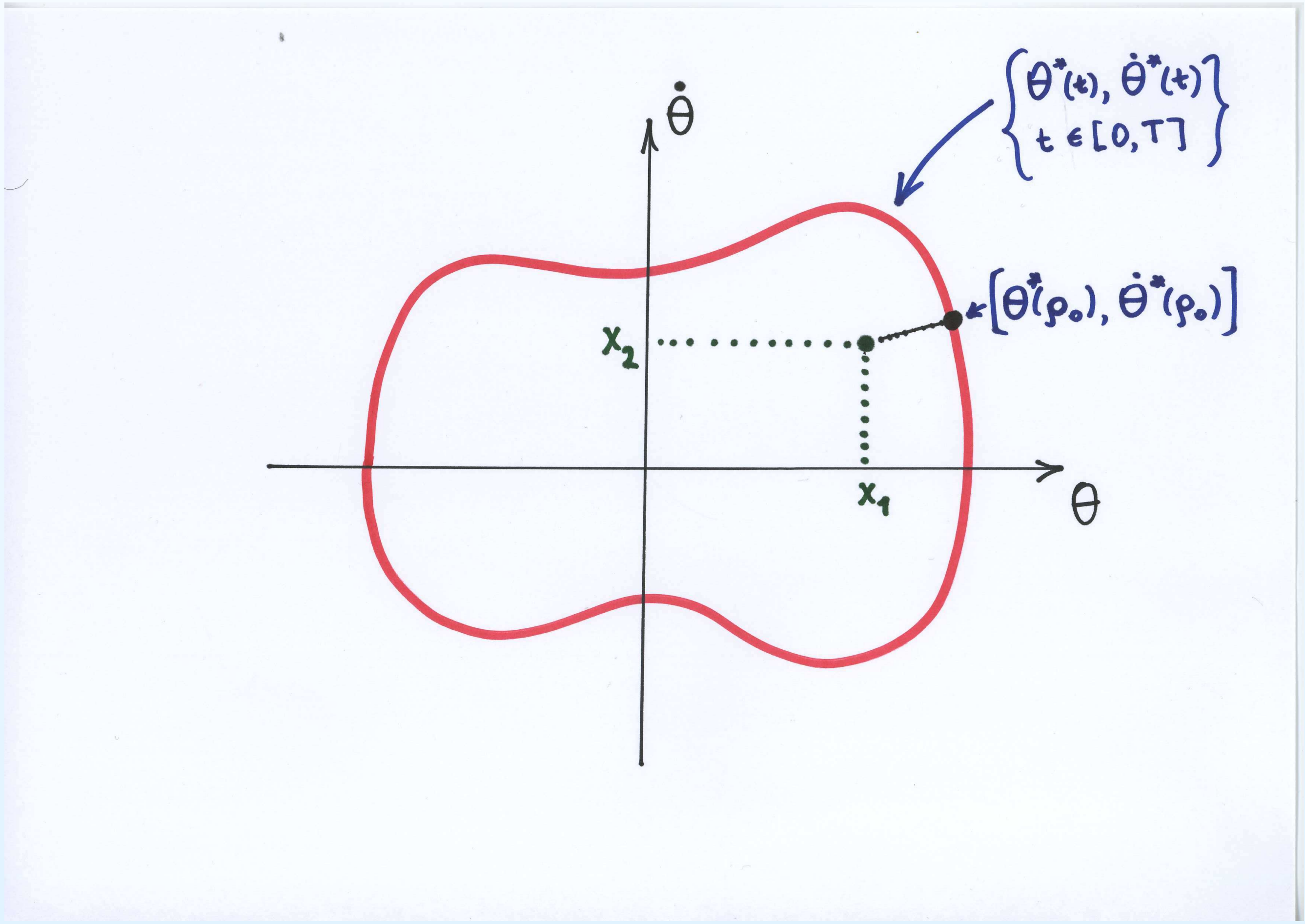
preserves its zero-value along this (even unbounded) solution

$$I\left(\theta(t), \dot{\theta}(t), \theta_0, \dot{\theta}_0\right) \equiv 0$$

Integral is a Distance: *Given the target orbit* $[\theta^\star(t), \dot\theta^\star(t)]$, *then*

- For any x_1 and x_2 the function $I(\cdot)$ satisfies the identity

$$I\left(x_1, x_2, \theta^\star(0), \dot\theta^\star(0)\right) \equiv I\left(x_1, x_2, \theta^\star(\rho), \dot\theta^\star(\rho)\right), \quad \rho \in [0, T]$$



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- Nearby the target orbit $|I(\cdot)|$ measures the distance to the orbit. Namely, the following approximation holds

$$I\left(x_1, x_2, \theta^\star(\rho_0), \dot\theta^\star(\rho_0)\right)^2 =$$

$$= \min_{0\le\rho<T}\left\{|x_1 - \theta^\star(\rho)|^2 + |x_2 - \dot\theta^\star(\rho)|^2\right\} \times$$

$$\times 4\left[\dot\theta^\star(\rho_0)^2 + \ddot\theta^\star(\rho_0)^2\right] + \ldots$$

Here

$$\rho_0 = \arg\min_{0\le\rho<T}\left\{|x_1 - \theta^\star(\rho)|^2 + |x_2 - \dot\theta^\star(\rho)|^2\right\}$$

New Passivity Relation:

The time derivative of the function $I(\theta, \dot\theta, x, y)$ *defined as*

$$I = \dot\theta^2 - \exp\left\{-2\int_x^\theta \frac{\beta(\tau)}{\alpha(\tau)}d\tau\right\}\left[y^2 - \int_x^\theta \exp\left\{2\int_x^s \frac{\beta(\tau)}{\alpha(\tau)}d\tau\right\}\frac{2\gamma(s)}{\alpha(s)}ds\right]$$

with x *and* y *being some constants, calculated along a solution of the system*

$$\alpha(\theta)\ddot\theta + \beta(\theta)\dot\theta^2 + \gamma(\theta) = W$$

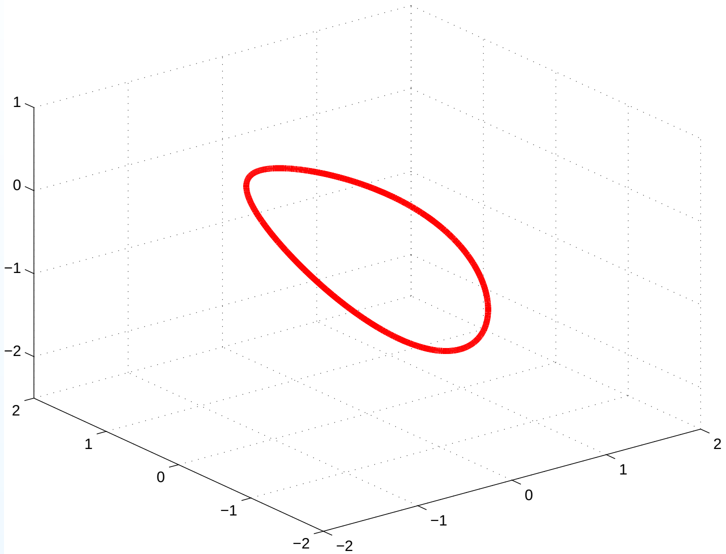
has the form

$$\frac{d}{dt}I(\theta, \dot\theta, x, y) = \dot\theta\left\{\frac{2}{\alpha(\theta)}W - \frac{2\beta(\theta)}{\alpha(\theta)}I(\theta, \dot\theta, x, y)\right\}$$

Outline

- Motivation and Preliminaries
- Representation of a Motion for a Mechanical System
- Good Coordinates around a Target Motion
- Example

Transverse Coordinates:



Given a T-periodic motion

$$q^\star(t) = \Big(q_1^\star(t),\ q_2^\star(t),\ \ldots,\ q_n^\star(t)\Big)^T$$

There are n-functions

$$\phi_1(\cdot),\ \phi_2(\cdot),\ \ldots,\ \phi_n(\cdot)$$

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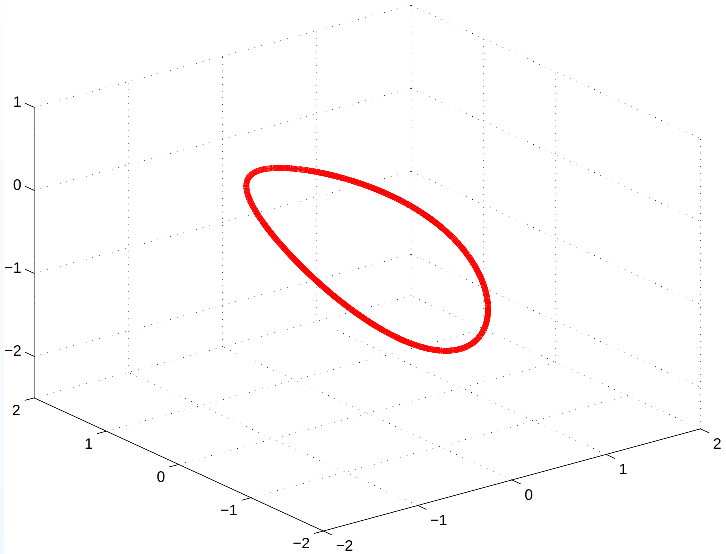

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We can always assume that $q_n = \theta \quad \Rightarrow \quad \phi_n(\cdot)$ is trivial!

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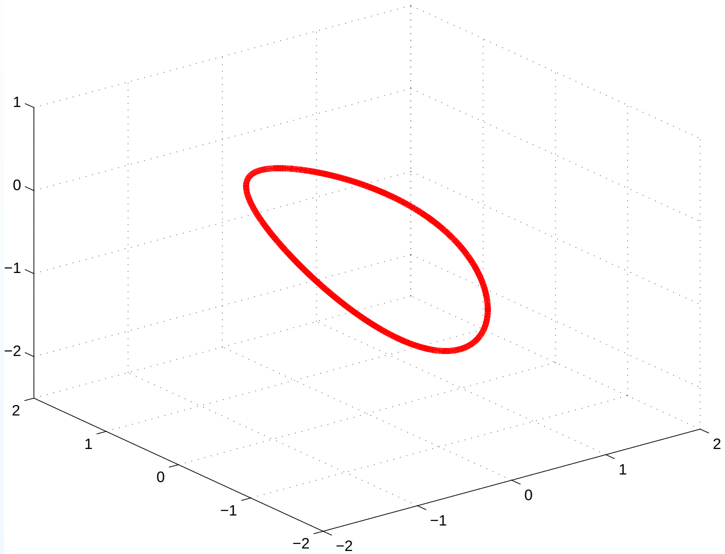

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There are n-functions

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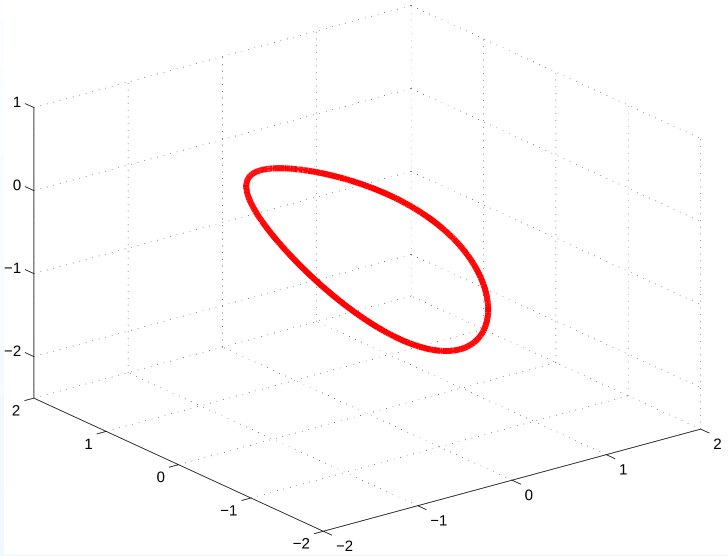
We can always assume that $q_n = \theta \quad \Rightarrow \quad \phi_n(\cdot)$ is trivial!

New n-generalized coordinates are θ and $y = \Big(y_1,\, \ldots,\, y_{n-1}\Big)$

$$y_1 = q_1 - \phi_1(\theta), \quad \ldots, \quad y_{n-1} = q_{n-1} - \phi_{n-1}(\theta)$$

Transverse Coordinates:



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New n-generalized coordinates are θ and $y = \left(y_1,\, \ldots,\, y_{n-1}\right)$

$$y_1 = q_1 - \phi_1(\theta), \quad \ldots, \quad y_{n-1} = q_{n-1} - \phi_{n-1}(\theta)$$

For the E-L system $x = (q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n)^T$, $\dim x = 2n$

Consider as a candidate for $x_\perp$-variable the set of quantities

$$x_\perp = \left[I(\theta, \dot{\theta}, \theta^\star(0), \dot{\theta}^\star(0)),\, y_1,\, \ldots,\, y_{n-1},\, \dot{y}_1,\, \ldots,\, \dot{y}_{n-1}\right]^T$$

Transverse Coordinates:

With the choice

$$x_{\perp} = \Big[I(\theta, \dot{\theta}, \theta^{\star}(0), \dot{\theta}^{\star}(0)),\; y_1,\; \ldots,\; y_{n-1},\; \dot{y}_1,\; \ldots,\; \dot{y}_{n-1} \Big]^T$$

one can compute analytically the linearization of transverse dynamics of the underactuated Euler-Lagrange system

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\right] - \frac{\partial \mathcal{L}}{\partial q} = R(q, \dot{q}) + B(q)u$$

around its solution

$$q^{\star}(t) = \Big(q_1^{\star}(t),\; q_2^{\star}(t),\; \ldots,\; q_n^{\star}(t) \Big)^T$$

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How to Plan a Motion for a Unicycle

Motion planning for the dynamical model

$$\begin{aligned} \ddot{x} &= -\left[\dot{y}\cdot\sin\theta + \dot{x}\cdot\cos\theta\right]\cdot\dot{\theta}\cdot\sin(\theta) \\ \ddot{y} &= \left[\dot{y}\cdot\sin\theta + \dot{x}\cdot\cos\theta\right]\cdot\dot{\theta}\cdot\cos(\theta) \\ J\ddot{\theta} &= u \end{aligned}$$

can be quite non-trivial.

How to Plan a Motion for a Unicycle

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$$
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\ddot{y} &= [\dot{y} \cdot \sin\theta + \dot{x} \cdot \cos\theta] \cdot \dot{\theta} \cdot \cos(\theta) \\
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\end{aligned}
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Find feasible motions of the system consistent with requirement

the center of mass should stay on a circle of radius R

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can be quite non-trivial.

Find feasible motions of the system consistent with requirement

the center of mass should stay on a circle of radius R

I.e. along any such motion $[x_c(t), y_c(t), \theta_c(t)]$ the relations hold

$$x_c(t) = R\cdot\cos\left(\theta_c(t) - \frac{\pi}{2}\right) \qquad y_c(t) = R\cdot\sin\left(\theta_c(t) - \frac{\pi}{2}\right)$$

How to Plan a Motion for a Unicycle

Motion planning for the dynamical model

$$\begin{aligned} \ddot{x} &= -\left[\dot{y}\cdot\sin\theta + \dot{x}\cdot\cos\theta\right]\cdot\dot{\theta}\cdot\sin(\theta) \\ \ddot{y} &= \left[\dot{y}\cdot\sin\theta + \dot{x}\cdot\cos\theta\right]\cdot\dot{\theta}\cdot\cos(\theta) \\ J\ddot{\theta} &= u \end{aligned}$$

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$$\ddot{x}_c = R\left[\cos\theta_c\ddot{\theta}_c - \sin\theta_c\dot{\theta}_c^2\right]$$

$$\ddot{y}_c = R\left[\sin\theta_c\ddot{\theta}_c + \cos\theta_c\dot{\theta}_c^2\right]$$

How to Plan a Motion for a Unicycle

Along a circular motion $[x_c(t), y_c(t), \theta_c(t)]$ of the system

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$$

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$$\Downarrow$$

$$\cos\theta_c \cdot \ddot{x}_c + \sin\theta_c \cdot \ddot{y}_c = R\ddot{\theta}_c$$

How to Plan a Motion for a Unicycle

Along a circular motion $[x_c(t), y_c(t), \theta_c(t)]$ of the system

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How to Plan a Motion for a Unicycle

Any circular motion $[x_c(t), y_c(t), \theta_c(t)]$ of the system

$$
\begin{aligned}
\ddot{x} &= -[\dot{y} \cdot \sin\theta + \dot{x} \cdot \cos\theta] \cdot \dot{\theta} \cdot \sin(\theta) \\
\ddot{y} &= [\dot{y} \cdot \sin\theta + \dot{x} \cdot \cos\theta] \cdot \dot{\theta} \cdot \cos(\theta) \\
J\ddot{\theta} &= u
\end{aligned}
$$

has the form

$$
\begin{aligned}
\theta_c(t) &= \omega_c \cdot t + \theta_0 \\
x_c(t) &= R \cdot \sin\theta_c(t) \\
y_c(t) &= -R \cdot \cos\theta_c(t) \\
u_c(t) &= 0
\end{aligned}
$$

Orbital stability of the motion means that the distance to the trajectory decays to zero

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The state space vector of the mechanical system is

$$\left[x,\, y,\, \theta,\, \dot{x},\, \dot{y},\, \dot{\theta} \right]$$

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We search for change of coordinates

$$\left[\, \bullet\,,\; \bullet\,,\; \bullet\,,\; \bullet\,,\; \bullet\,,\; \bullet \,\right]$$

such that most of new coordinates equal to ***zero*** on the motion

$$\theta_\star(t) = \omega \cdot t + \theta_0, \quad x_\star(t) = R \sin(\omega \cdot t + \theta_0), \quad y_\star(t) = -R \cos(\omega \cdot t + \theta_0)$$

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Candidates

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$$\dot{z}_1 = \dot{x} - R\cos\theta\cdot\dot{\theta}, \qquad \dot{z}_2 = \dot{y} - R\sin\theta\cdot\dot{\theta}$$

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The dynamics of $[\theta, \dot{\theta}]$-variables

$$J\ddot{\theta} = u$$

can be rewritten in $[\theta, I]$-coordinates with $I = \dot{\theta}^2 - \omega^2$

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such that most of new coordinates equal to $zero$ on the motion

Candidates for transverse coordinates

$$z_1 = x - R\sin\theta, \quad z_2 = y + R\cos\theta, \quad I = \dot{\theta}^2 - \omega^2$$

$$\dot{z}_1 = \dot{x} - R\cos\theta\cdot\dot{\theta}, \quad \dot{z}_2 = \dot{y} - R\sin\theta\cdot\dot{\theta}$$

Steps in Orbital Stabilization of a Cyclic Motion of a Coin

Linearization of transverse coordinates

$$X_\perp = [\, z_1,\, z_2,\, \dot{z}_1,\, \dot{z}_2,\, I \,]$$

along the motion

$$\theta_\star(t) = \omega \cdot t + \theta_0, \quad x_\star(t) = R \sin(\omega \cdot t + \theta_0), \quad y_\star(t) = -R \cos(\omega \cdot t + \theta_0)$$

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has the form

$$\frac{d}{dt}[\delta X] = A(t)\delta X + B(t)\delta u$$

Steps in Orbital Stabilization of a Cyclic Motion of a Coin

Coefficients of $A(t)$ and $B(t)$ are

$$A(t) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\omega & 0 \\ 0 & 0 & \omega & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B(t) = \begin{bmatrix} 0 \\ 0 \\ -R\cos(\omega t + \theta_0) \\ -R\sin(\omega t + \theta_0) \\ \omega / J \end{bmatrix}$$

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The controllability Gramian computed over the period

$$W_c = \int_0^{2\pi/\omega} e^{-A\tau} B(\tau) B(\tau)^T e^{-A^T\tau} d\tau$$

has three positive and two zero eigenvalues for any $J,\ R > 0$

$$\lambda(W_c) = \{w_1,\ w_2,\ w_3,\ 0,\ 0\}, \quad w_i > 0$$

Concluding Remarks:

- We suggest a choice of transverse coordinates for a motion of mechanical system, which experience the quadratic in velocities reaction forces;

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  - Procedure for synthesis of orbitally stabilizing controller
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- The approach can be extended if the motion does not admit parametrization by one choice of VHC