Anytime Reliability of Systematic...

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Anytime Reliability of Systematic LDPC Convolutional Codes

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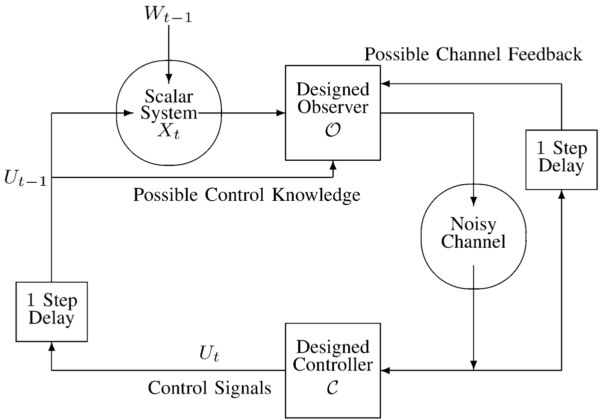
Overview

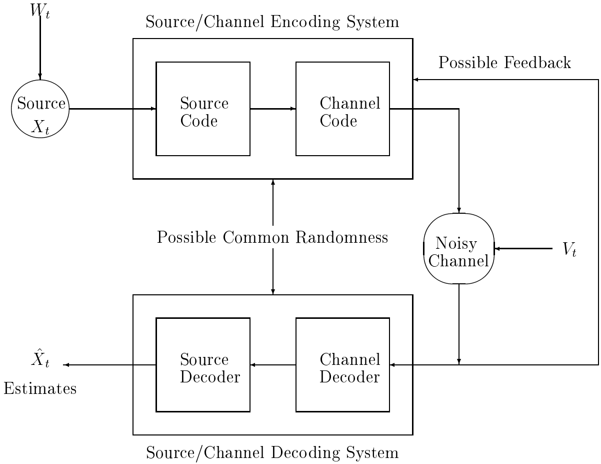
Overview

Automatic Control over Noisy Channels



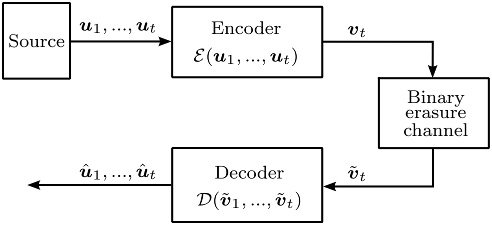
A. Sahai and S. Mitter, "The necessity and sufficiency of anytime capacity for stabilization of a linear system over a noisy communication link - Part I: Scalar systems," *IEEE Trans. Inf. Theory*, vol. 52, no. 8, pp. 3369–3395, Aug. 2006.

Model for Anytime Communications



A. Sahai, "Anytime information theory," Ph.D. dissertation, MIT, 2001.

Model for Anytime Channel-Coded Transmission



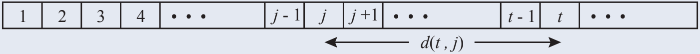
Encoding and Decoding

$$\mathbf{u}_{[1,t]} = [\mathbf{u}_1, \mathbf{u}_2 \ldots, \mathbf{u}_t]$$
$$\mathbf{v}_t = \mathcal{E}(\mathbf{u}_1, \ldots, \mathbf{u}_t)$$
$$\hat{\mathbf{u}}_{[1,t]} = [\hat{\mathbf{u}}_1, \ldots, \hat{\mathbf{u}}_t] = \mathcal{D}(\tilde{\mathbf{v}}_1, \ldots, \tilde{\mathbf{v}}_t)$$

Anytime Reliability

Anytime Reliability

- The receiver can decide to start decoding at *anytime*



- Anytime reliability can formally be defined as

$$P(\hat{\mathbf{u}}_j \neq \mathbf{u}_j | \mathbf{u}_{[1,t]} \text{was transmitted}) \leq \beta 2^{-\alpha d(t,j)} \quad (1)$$

- For a particular code at rate R, the largest α such that (1) is fulfilled is referred to as the *anytime exponent* of the code

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Motivation

LDPC Convolutional Codes

Anytime LDPC Convolutional Codes

Asymptotic Analysis

Numerical Examples

Summary and Concluding Remarks

Selected Prior Work

- A. Sahai, "Anytime information theory," Ph.D. dissertation, MIT, 2001.
- A. Sahai and S. Mitter, "The necessity and sufficiency of anytime capacity for stabilization of a linear system over a noisy communication link - Part I: Scalar systems," *IEEE Trans. Inf. Theory*, vol. 52, no. 8, pp. 3369–3395, Aug. 2006.
- L. J. Schulman, "Coding for interactive communication," *IEEE Trans. Inf. Theory*, vol. 42, no. 6, pp. 1745–1756, Jun. 1996.
- R. Ostrovsky, Y. Rabani, and L. J. Schulman, "Error-correcting codes for automatic control," *IEEE Trans. Inf. Theory*, vol. 55, no. 7, pp. 2931–2941, Jul. 2009.
- G. Como, F. Fagnani, and S. Zampieri, "Anytime reliable transmission of real-valued information through digital noisy channels," *SIAM J. Control and Opt.*, vol. 48, no. 6, pp. 3903–3924, Mar. 2010.
- R. T. Sukhavasi and B. Hassibi, "Linear error correcting codes with anytime reliability," in *IEEE Int. Symp. Inf. Theory*, St. Petersburg, Rusia, Jun. 2011.

Our Contributions

Anytime LDPC Convolutional Codes

- Modern coding structures have not yet been considered for anytime transmission
- We propose:
  - a tractable protograph structure for an LDPC-CC ensemble
  - an expanding-window decoding scheme
- We show that the ensemble asymptotically exhibits the desired anytime properties
- We show through simulation that the ensemble also exhibits some anytime properties for finite-length codes

Overview

LDPC Convolutional Codes

Background

- Invented in

  A. J. Felström and K. Sh. Zigangirov, "Time-varying periodic convolutional codes with low-density parity-check matrix," *IEEE Trans. on Inf. Theory*, vol. 45, no. 6, pp. 2181–2191, Sept. 1999.

- Good performance has been analysed in

  M. Lentmaier, A. Sridharan, D. J. Costello, and K. Sh. Zigangirov, "Iterative decoding threshold analysis for LDPC convolutional codes," *IEEE Trans. on Inf. Theory*, vol. 56, no. 10, pp. 5274 – 5289, Oct. 2010.

  $\Rightarrow$ *"For a terminated LDPCC code ensemble, the thresholds are better than for corresponding regular and irregular LDPC block codes"*

- Capacity achieving property has been proven in

  S. Kudekar, T. Richardson, and R. Urbanke, "Threshold saturation via spatial coupling: Why convolutional LDPC ensembles perform so well over the BEC," *IEEE Trans. on Inf. Theory*, vol. 57, no. 2, pp. 803 – 834, Feb. 2011.

  $\Rightarrow$ *"Spatial coupling of individual codes increases the belief-propagation (BP) threshold of the new ensemble to its maximum possible value, namely the maximum a posteriori (MAP) threshold of the underlying ensemble."*

- Implementation aspects

  A. E. Pusane, A. J. Felström, A. Sridharan, M. Lentmaier, K. Sh. Zigangirov, and D. J. Costello, "Implementation aspects of LDPC convolutional codes," *IEEE Trans. on Comm.*, vol. 56, no. 7, pp. 1060 – 1069, July 2008.

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LDPC Convolutional Codes

- A rate $R = b/c$ LDPC convolutional code is defined as a set of sequences $\mathbf{v}_{[0,L-1]} = [\mathbf{v}_0, \ldots, \mathbf{v}_{L-1}]$ that satisfy

$$\mathbf{0} = \mathbf{v}_{[0,L-1]}\mathbf{H}^T_{[0,L-1]} =$$

$$\mathbf{v}_{[0,L-1]} \underbrace{\begin{bmatrix} \mathbf{H}_0^T(0) & \ldots & \mathbf{H}_{m_s}^T(m_s) & & \\ & \mathbf{H}_0^T(1) & \ldots & \mathbf{H}_{m_s}^T(m_s+1) & \\ & \ddots & & \ddots & \\ & & \mathbf{H}_0^T(L-1-m_s) & \ldots & \mathbf{H}_{m_s}^T(L-1) \end{bmatrix}}_{\mathbf{H}^T_{[0,L-1]}}$$

where

- $\mathbf{H}^T_{[0,L-1]}(t)$ is the *syndrome former matrix* (i.e., the transposed parity check matrix $\mathbf{H}_{[0,L-1]}$),
- $\mathbf{H}_i^T(t)$ is a $c \times (c-b)$ binary matrix,
- $\mathbf{H}_0^T(t)$ must have full rank $\forall t$,
- L is the number of positions; length of the code: cL
- m_s is the syndrome former memory.

- For LDPC-CCs the syndrome former matrix is sparse.

LDPC Convolutional Codes

$(J, K = \kappa J, L, M)$ regular LDPC convolutional code ensemble

- Syndrome former memory: $m_s = J - 1$
- Submatrices

$$\mathbf{H}_i^T(t) = [\mathbf{P}_i^{(1)}(t), \ldots, \mathbf{P}_i^{(\kappa)}(t)]^T,$$

  with $M \times M$ permutation matrices $\mathbf{P}_j(t)$.
- Example: $J = 3,\ \kappa = 2,\ K = 6,\ m_s = 2$

$$\mathbf{H}_{[0,L]}^T = \begin{bmatrix} \mathbf{P}_0^{T(1)}(0) & \mathbf{P}_1^{T(1)}(1) & \mathbf{P}_2^{T(1)}(2) & & & \\ \mathbf{P}_0^{T(2)}(0) & \mathbf{P}_1^{T(2)}(1) & \mathbf{P}_2^{T(2)}(2) & & & \\ & \mathbf{P}_0^{T(1)}(1) & \mathbf{P}_1^{T(1)}(2) & \mathbf{P}_2^{T(1)}(3) & & \\ & \mathbf{P}_0^{T(2)}(1) & \mathbf{P}_1^{T(2)}(2) & \mathbf{P}_2^{T(2)}(3) & & \\ & & \mathbf{P}_0^{T(1)}(2) & \mathbf{P}_1^{T(1)}(3) & \mathbf{P}_2^{T(1)}(4) & \\ & & \mathbf{P}_0^{T(2)}(2) & \mathbf{P}_1^{T(2)}(3) & \mathbf{P}_2^{T(2)}(4) & \\ & & & & & \ddots \end{bmatrix}$$

- Rate $R = 1 - J/K = 1 - 1/\kappa$ (not considering rate loss due to initialization and termination)

Protograph Representation

$(J, K = \kappa J, L, M)$ regular LDPC convolutional code ensemble

- Example: $J = 3,\ \kappa = 2,\ K = 6,\ m_s = 2$

$$\mathbf{B}_{[0,L]}^T = \begin{bmatrix} \mathbf{B}_0^T(0) & \mathbf{B}_1^T(1) & \mathbf{B}_2^T(2) & & & \\ & \mathbf{B}_0^T(1) & \mathbf{B}_1^T(2) & \mathbf{B}_2^T(3) & & \\ & & \mathbf{B}_0^T(2) & \mathbf{B}_1^T(3) & \mathbf{B}_2^T(4) & \\ & & & & & \ddots \end{bmatrix}$$

where

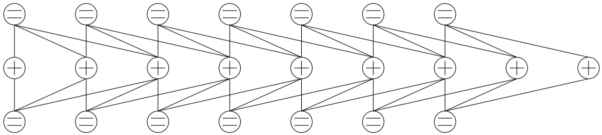
$$\mathbf{B}_i^T(t) = [1, 1]^T,$$

- Each 1 element in the protograph is then "lifted" with an $M \times M$ randomized permutation matrices $\mathbf{P}(t)$.

Protograph Representation

$(J, K = \kappa J, M)$ regular LDPC convolutional code ensemble

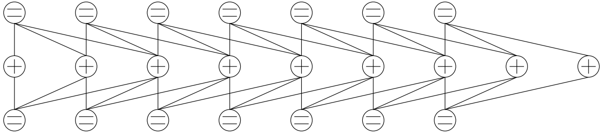
- Protograph: $J = 3$, $K = 6$, $m_s = 2$



- Observation: irregular check degrees at the boundaries; regular variable node degrees.
- The good performance relies on this property!
- Decoding is done with iterative message-passing over the code graph for terminated codes
- Sliding-window message-passing is possible for non-terminated codes

Multi-Edge Density Evolution

- Densities have to be evaluated for each edge in the protograph (similar to iterative decoding on the protograph).



  - Variable nodes at position t are connected to check nodes at positions $t, \dots, t + m_s$
  - Check nodes at position t are connected to check nodes at positions $t - m_s, \dots, t$

- Density evolution for the BEC case:
  - Erasure probability for messages from variable nodes at position t to check nodes at position $t + j$ (i-th iteration):

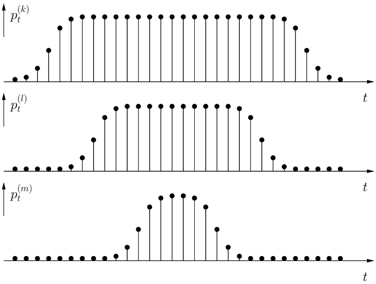
$$p^{(i)}_{t,t+j} = \epsilon \prod_{k \neq j} q^{(i)}_{t,t+k}$$

  - Erasure probability for messages from check nodes at position t to variable nodes at position $t - j$ (i-th iteration):

$$q^{(i)}_{t,t-j} = 1 - (1 - p^{(i-1)}_{t-j,t})^{\kappa-1} \prod_{k \neq j} (1 - p^{(i-1)}_{t-k,t})^{\kappa}$$

Convergence Behavior

- Convergence starts at the boundaries and propagates towards the middle of the block.



- Positions at the boundaries benefit from the lower (locally irregular) check-node degrees.
- After decoding a position at the boundary, the nodes can be removed from the graph and the same irregular degree distribution is reproduced at the new boundary.

Overview

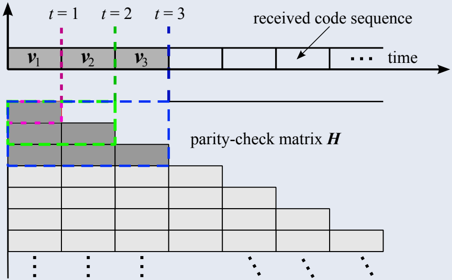
Anytime LDPC Convolutional Codes

Proposed Protograph Structure and Decoding Scheme

- Anytime linear code must have lower-left block-triangular structure

$$\mathbf{B}_{[1,t]} = \begin{bmatrix} \mathbf{B}_0 & & & & \\ \mathbf{B}_1 & \mathbf{B}_0 & & & \\ \vdots & \mathbf{B}_1 & \ddots & & \\ \vdots & \vdots & \ddots & & \\ \mathbf{B}_{t-1} & \mathbf{B}_{t-2} & \dots & \mathbf{B}_1 & \mathbf{B}_0 \end{bmatrix}.$$

- Expanding-window decoder



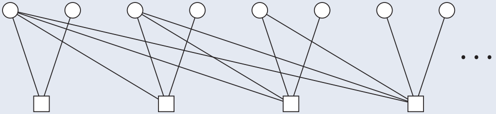
Example Ensemble for Consideration

Tractable Protograph Structure

- Ensemble of regular and systematic protographs of rate $R = 1/2$

$$\mathbf{B}_{[1,t]} = \begin{bmatrix} 1 & 1 & & & & & & & \\ 1 & 0 & 1 & 1 & & & & & \\ 1 & 0 & 1 & 0 & 1 & 1 & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & \\ 1 & 0 & 1 & 0 & 1 & 0 & \dots & 1 & 1 \end{bmatrix},$$

- For ease of analysis we set $\mathbf{B}_0 = [1\ 1]$ and $\mathbf{B}_i = [1\ 0]$ for $i \neq 0$
- The structure of the factor graph is as follows

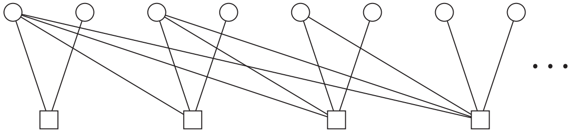


Overview

Asymptotic Analysis

P-EXIT Analysis

- Asymptotic erasure performance over time in the limit of
  - infinite block size ($M \to \infty$), and
  - infinite number of decoder iterations ($k \to \infty$)
- The recursive expression are
  - C-to-V node: $I_{Av,t}^{k+1}(i,j) = \prod_{s=1,s\neq j}^{2i} I_{Ev,t}^{k}(i,s)$
  - V-to-C node: $I_{Ev,t}^{k+1}(i,j) = 1 - \epsilon \prod_{s=\lceil j/2 \rceil, s\neq i}^{t} (1 - I_{Av,t}^{k+1}(s,j))$
  - APP-LLR : $I_{\mathrm{APP},t}(j) = 1 - \epsilon \prod_{s=\lceil j/2 \rceil}^{t} (1 - I_{Av}^{\infty}(s,j))$



G. Liva and M. Chiani, "Protograph LDPC codes design based on EXIT analysis," in *IEEE Global Telecom. Conf.*, Washington D. C., USA, Nov. 2007, pp. 3250–3254.

Asymptotic Analysis

Main Result 1

For $M \to \infty$ and $k \to \infty$

$$P_{\mathrm{APP},t}(j) = P_{\mathrm{APP},t}(j+2)\ \epsilon$$

so performance curves a shifted versions of the same curve

Main Result 2

For $M \to \infty$, $k \to \infty$, increasing t and small j relative to t

$$P_{\mathrm{APP},t+1}(j) = P_{\mathrm{APP},t}(j)\ \epsilon$$

leading to an anytime exponent of $\alpha = -\log \epsilon$

Asymptotic Analysis

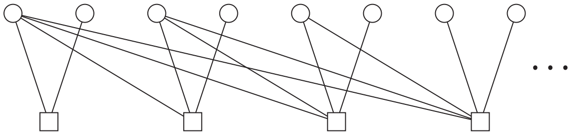
Asymptotic Relationships Applied in the Proofs

$$I^k_{Av,t}(i+1,j) \leq I^k_{Av,t}(i,j)$$

$$I^k_{Ev,t}(i,j+j') \leq I^k_{Ev,t}(i,j), \text{ for } j, j+j' \text{ odd}$$

$$I^k_{Ev,t}(i,j) = 1-\epsilon \quad \forall k, \text{ for } j \text{ even and } j = 2t-1$$

$$I^\infty_{Av,t}(\lceil j/2 \rceil, j) = 1-\epsilon, \text{ for odd and small } j \text{ relative to } t$$



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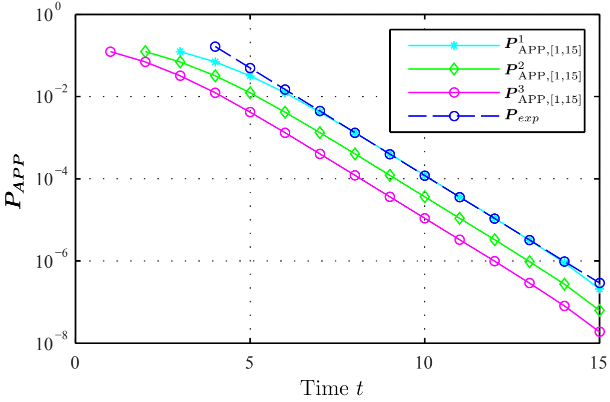
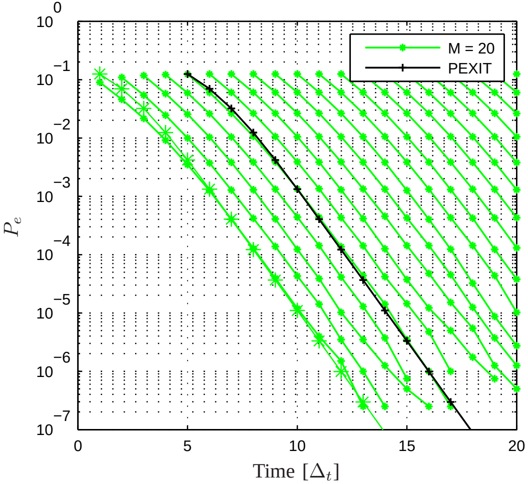
Asymptotic Decoding Erasure Probability

- Erasure probability $\epsilon = 0.3$
- Performance of decoding blocks over time



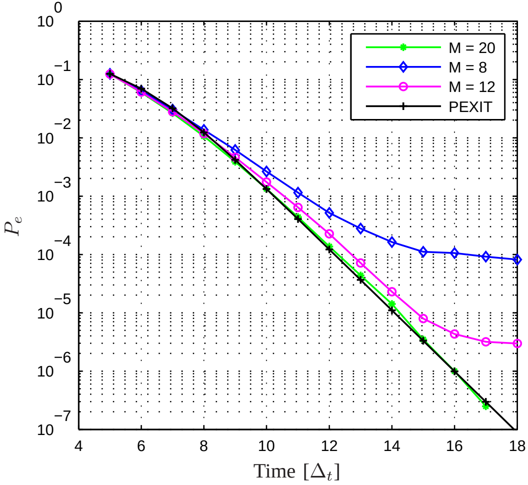
Asymptotic Decoding Erasure Probability

- Comparison to finite-length case of $M = 20$ over time



Asymptotic Decoding Erasure Probability

- Comparison to finite-length cases of $M = 8, 12, 20$ over time



Overview

Summary and Concluding Remarks

Summary

- Investigated a particular ensemble of anytime LDPC convolutional codes
- Showed that anytime reliability is asymptotically achieved as block length and number of iterations grow large
- Compared favorably with finite-length simulation results

Concluding Remarks

- A regular systematic anytime LDPC CC achieves anytime reliability
- Block length does not need to grow large to achieve anytime reliability
- Irregular systematic anytime LDPC CCs have potential for better performance
- We are currently developing analysis techniques for finite-length anytime LDPC CCs