Linear State Estimation Via Multiple Sensors Over Rate-Constrained Channels

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Outline

- Introduction & Motivation
- Multi-terminal estimation problems
- Single Sensor
- Multiple Sensors
- Numerical Studies
- Remarks and Conclusions

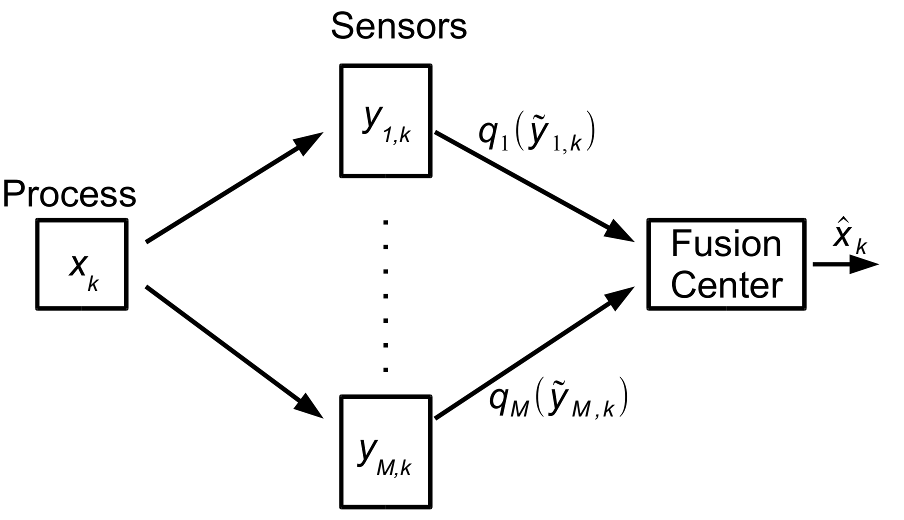
Introduction

- Linear state estimation using multiple sensors is a commonly performed task in e.g. radar tracking, industrial monitoring, remote sensing, wireless control systems, mobile robotics

- Many systems nowadays use digital communications
  - Analog signals need to be quantized
- Wireless channels are bandwidth limited
  - Sensor network applications: severe bandwidth limitations

- Characterize the trade-off between estimation performance and quantization rate (extension of the traditional rate-distortion theory)

Introduction



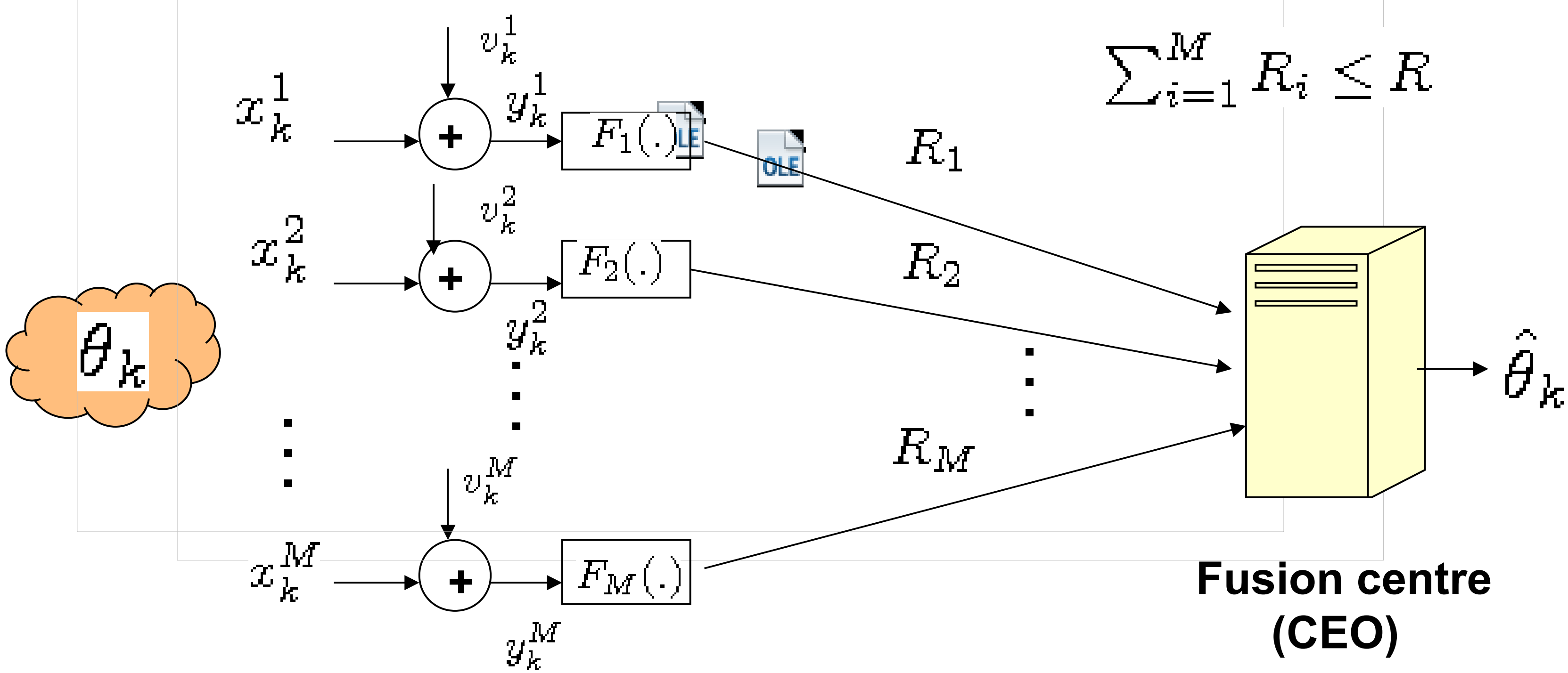
- Estimate a discrete time linear system
- Sensors transmit quantized innovations
  - For unstable systems, states become unbounded while innovations remains of bounded variance
- In our work, we establish a relationship between quantization rate and estimation error for linear dynamical system in a multi-terminal setting, in the case of high rate quantization

Introduction – Related Work

- Similar ideas of quantizing the innovations have been previously considered
  - [Nair&Evans,04] - Single sensor, stable scheme but performance difficult to analyze
  - [Msechu et al. 2008], [You et al. 2011] – Estimator not stable for unstable systems
  - [Sukhavasi and Hassibi, 2011] – Single sensor, particle filter based scheme, performance difficult to analyze
  - [Fu and deSouza, 2009] – Single sensor, logarithmic quantizer, proof of stability for bounded noise
  - Information theoretic multi-terminal estimation: CEO problem

The CEO Problem

- *Simplified single-hop setup:* multiple sensors communicating with a fusion centre over bandwidth constrained channels



The CEO Problem

- Original Results: Viswanathan and Berger [1996] for an i.i.d. scalar Gaussian source
- Rate distortion region: Oohama [1998] for an i.i.d. scalar Gaussian source
- Recent extensions to vector sources and correlated noise across sensors
- Most of these results apply to memoryless sources (at most stationary) and require source coding over asymptotically large block lengths
- Cannot be applied to linear dynamical systems (which have memory and may be unstable) or systems where coding over large numbers of blocks may not be feasible (delay-sensitive applications e.g. wireless control)

Multi-terminal state estimation for linear dynamical systems with rate constraints

- Basic ideas: quantize the innovations (requires smart sensors who can perform their own Kalman filtering) at each sensor
- Apply high rate quantization theory (although in theory this only applies at high rates, performance is quite good at moderate rates (3-4 bits per sample))
- We will study the single sensor case first, followed by multiple sensors
- Difficulty: static quantization may not result in a stable estimate for unstable systems, need to use dynamic quantization
- Assumption: Fusion centre has knowledge of system parameters

Single Sensor

- Vector system $x_{k+1} = Ax_k + w_k$
- Scalar sensor measurement $y_k = Cx_k + v_k$

- Without quantization, optimal estimation given by Kalman filter

$$\begin{aligned}
\hat{x}^{kf}_{k|k-1} &= A\hat{x}^{kf}_{k-1|k-1} \\
\hat{x}^{kf}_{k|k} &= \hat{x}^{kf}_{k|k-1} + K^{kf}_k(y_k - C\hat{x}^{kf}_{k|k-1}) = \hat{x}^{kf}_{k|k-1} + K^{kf}_k \tilde{y}^{kf}_k \\
K^{kf}_k &= P^{kf}_{k|k-1}C^T(CP^{kf}_{k|k-1}C^T + \Sigma_v)^{-1} \\
P^{kf}_{k|k-1} &= AP^{kf}_{k-1|k-1}A^T + \Sigma_w \\
P^{kf}_{k|k} &= P^{kf}_{k|k-1} - P^{kf}_{k|k-1}C^T(CP^{kf}_{k|k-1}C^T + \Sigma_v)^{-1}CP^{kf}_{k|k-1}
\end{aligned}$$

- Innovations process $\tilde{y}^{kf}_k \triangleq y_k - C\hat{x}^{kf}_{k|k-1} \sim N(0, CP^{kf}_{k|k-1}C^T + \Sigma_v)$

Single Sensor – Quantized Filtering Scheme

- Quantized filtering scheme (at both sensor and fusion centre)

$$
\begin{aligned}
\hat{x}_{k|k-1} &= A\hat{x}_{k-1|k-1} \\
\hat{x}_{k|k} &= \hat{x}_{k|k-1} + K_k l_k q\left(\frac{y_k - C\hat{x}_{k|k-1}}{l_k}\right) \\
K_k &= P_{k|k-1}C^T(CP_{k|k-1}C^T + \Sigma_v + \Sigma_{n,k})^{-1} \\
P_{k|k-1} &= AP_{k-1|k-1}A^T + \Sigma_w \\
P_{k|k} &= P_{k|k-1} - P_{k|k-1}C^T(CP_{k|k-1}C^T + \Sigma_v + \Sigma_{n,k})^{-1}CP_{k|k-1}
\end{aligned}
$$

- $l_k q(\frac{y_k - c\hat{x}_{k|k-1}}{l_k})$ is quantization of the "innovations"
- l_k is scaling factor for adaptive "zooming" quantizers
  - if quantizer saturates, can "zoom out"
  - used to prove stability for unbounded (Gaussian) noise
- $\Sigma_{n,k}$ is an extra term to account for quantization noise variance

Single Sensor – Quantized Filtering Scheme

- Use shorthand $P_k = P_{k|k-1}$
- Assume $y_k - C\hat{x}_{k|k-1}$ is approximately $N(0, CP_kC^T + \Sigma_v)$
- Can use a uniform quantizer of *N* levels
  - Asymptotically optimal quantizer range and distortion given in [Hui&Neuhoff,2001], can then obtain $\Sigma_{n,k} = \delta_N(CP_kC^T + \Sigma_v)$

    where $\delta_N = \frac{4 \ln N}{3N^2}$
  - Can generalize to lattice vector quantizers
- Can also use an "optimal" Lloyd-Max quantizer of *N* levels (optimal for Gaussian distribution)
  - Can obtain $\Sigma_{n,k} = \delta_N(CP_kC^T + \Sigma_v)$ where $\delta_N = \frac{\pi\sqrt{3}}{2N^2}$
  - Difficult to generalize to vector quantizers (optimal quantizers not known in general)
- Quantizer of [Nair&Evans,04] – Can be used but performance difficult to analyze

Single Sensor - Stability

- Choose

$$l_k = ||C||\tilde{l}_k + d_v$$

$$\tilde{l}_k = ||A(I - K_k C)||\tilde{l}_{k-1} + d_w + ||AK_k||d_v + ||AK_k||(||C||\tilde{l}_{k-1} + d_v)\kappa(\omega_{k-1})$$

with $d_w > 0$ and $d_v > 0$ being constants, and

$$\kappa(\omega_k) = \begin{cases} \frac{2\sqrt{\ln N}}{N} & , \text{ quantizer not saturated} \\ \sqrt{\ln(N)} & , \text{ quantizer saturated} \end{cases}$$

- Define $f_k = x_k - \hat{x}_{k|k-1}$

- Theorem:

$\mathbb{E}[||f_k||^2]$ is bounded $\forall k$ for sufficiently large N.

Single Sensor – Proof of Stability

- Sketch of proof

- Similar to [Nair&Evans,04], consider an upper bound to $\mathbb{E}[||f_k||^2]$ given by

$$||f_k, L||_* \triangleq \sqrt{\mathbb{E}[L^2 + |f_k|^{2+\epsilon} L^{-\epsilon}}$$

  for some random variable L>0 and some $\epsilon > 0$

- Can then show the following Lemma:

$$||X - Lq\left(\frac{X}{L}\right), L\kappa(\Omega)||_* \leq \frac{\zeta}{(\ln N)^{\epsilon/2}} ||X, L||_*$$

  where ζ is a constant that depends only on ϵ and N

Single Sensor – Proof of Stability

- Using the lemma and similar arguments from [Gurt&Nair,09], can then derive the recursive relationship

$$||f_{k+1}, \tilde{l}_{k+1}||_* \leq \left(||A(I - K_k C)|| + ||AK_k||.||C|| \frac{\zeta}{(\ln N)^{\epsilon/2}} \right) ||f_k, \tilde{l}_k||_*$$
$$+ ||w_k, d_w||_* + ||AK_k|| \left(1 + \frac{\zeta}{(\ln N)^{\epsilon/2}} \right) ||v_k, d_v||_*$$

- $||w_k, d_w||_*$ and $||v_k, d_v||_*$ can be upper bounded by constants
- Since $||A(I - K_k C)|| < 1$, and $K_k \to K$, we have

$$\left(||A(I - K_k C)|| + ||AK_k||.||C|| \frac{\zeta}{(\ln N)^{\epsilon/2}} \right) < 1$$

for *N* sufficiently large, which proves that $||f_k, \tilde{l}_k||_*$, and hence $\mathbb{E}[||f_k||^2]$, is bounded for all *k*

Single Sensor – Choice of scaling factors

- Recall

$$l_k = ||C||\tilde{l}_k + d_v$$

$$\tilde{l}_k = ||A(I - K_k C)||\tilde{l}_{k-1} + d_w + ||AK_k||d_v + ||AK_k||(||C||\tilde{l}_{k-1} + d_v)\kappa(\omega_{k-1})$$

- Choice of *dv* and *dw* can affect performance
- If we choose $d_v = \frac{1-||A(I-KC)||-||AK||.||C||\kappa_{min}-||C||d_w}{1-||A(I-KC)||+||AK||.||C||}$

  where *K* is the steady state value of *Kk* and $\kappa_{min} = \frac{2\sqrt{\ln N}}{N}$, then $l_k \approx 1$ for large *N*
  - Reason: For large *N*, quantizer saturation is rare. Choice of *dv* ensures that $l_k \to 1$ when saturation doesn't occur.

Single Sensor – Asymptotic Analysis

- *Pk* is an approximation to the mean squared error
- As $k \to \infty, \quad P_k \to P_\infty$ satisfying

$$P_\infty = AP_\infty A^T + \Sigma_w - \frac{AP_\infty C^T (CP_\infty C^T + \Sigma_v)^{-1} CP_\infty A^T}{1+\delta_N}$$

where $\delta_N = \begin{cases} \frac{\pi\sqrt{3}}{2N^2} & , \quad \text{optimal quantization} \\ \frac{4\ln N}{3N^2} & , \quad \text{optimal uniform quantization} \end{cases}$

- Assume high rate quantization (or large *N)* and analyze behaviour of P_∞ with *N*

- Difficulty - no closed form expression for P_∞ in vector systems

Single Sensor – Asymptotic Analysis

$$P_\infty = AP_\infty A^T + \Sigma_w - \frac{AP_\infty C^T (CP_\infty C^T + \Sigma_v)^{-1} CP_\infty A^T}{1+\delta_N}$$

- Technique used - Extend method for finding asymptotic solutions to algebraic equations in perturbation theory to matrices
- Write P_∞ as $P_\infty = \Phi_0 + \delta_N \Phi_1 + \delta_N^2 \Phi_2 + \ldots$
  where $\Phi_0, \Phi_1, \ldots$ are matrices not dependent on *N*
- Substitute $P_\infty = \Phi_0 + \delta_N \Phi_1 + \delta_N^2 \Phi_2 + \ldots$ into equation above

Single Sensor – Asymptotic Analysis

- Obtain

$$\Phi_0 + \delta_N \Phi_1 + \cdots = A(\Phi_0 + \delta_N \Phi_1 + \dots)A^T + \Sigma_w$$
$$-A(\Phi_0+\delta_N\Phi_1+\dots)C^T(C(\Phi_0+\delta_N\Phi_1+\dots)C^T+\Sigma_v)^{-1}$$
$$\times C(\Phi_0 + \delta_N \Phi_1 + \dots)A^T \frac{1}{1+\delta_N}$$
$$= A(\Phi_0+\delta_N\Phi_1+\dots)A^T+\Sigma_w - A(\Phi_0+\delta_N\Phi_i+\dots)C^T$$
$$\times [(C\Phi_0C^T + \Sigma_v)^{-1} - \delta_N(C\Phi_0C^T + \Sigma_v)^{-1}C\Phi_1C^T(C\Phi_0C^T + \Sigma_v)^{-1} + \dots]$$
$$\times C(\Phi_0 + \delta_N \Phi_1 + \dots)A^T(1 - \delta_N + \dots)$$

- Collect terms of same order to solve for $\Phi_0, \Phi_1, \dots$

Single Sensor – Asymptotic Analysis

- Collecting "constant" terms:
  $$\Phi_0 = A\Phi_0 A^T + \Sigma_w - A\Phi_0 C^T (C\Phi_0 C^T + \Sigma_v)^{-1} C\Phi_0 A^T$$
- Algebraic Riccati equation, can solve for Φ_0
- Same equation as satisfied by P_∞^{kf}, the steady state error covariance in the case of no quantization

- Collecting $O(\delta_N)$ terms:

$$\Phi_1 = \left(A - A\Phi_0 C^T (C\Phi_0 C^T + \Sigma_v)^{-1} C\right) \Phi_1 \left(A - A\Phi_0 C^T (C\Phi_0 C^T + \Sigma_v)^{-1} C\right)^T + A\Phi_0 C^T (C\Phi_0 C^T + \Sigma_v)^{-1} C\Phi_0 A^T$$

- Lyapunov equation, can solve for Φ_1

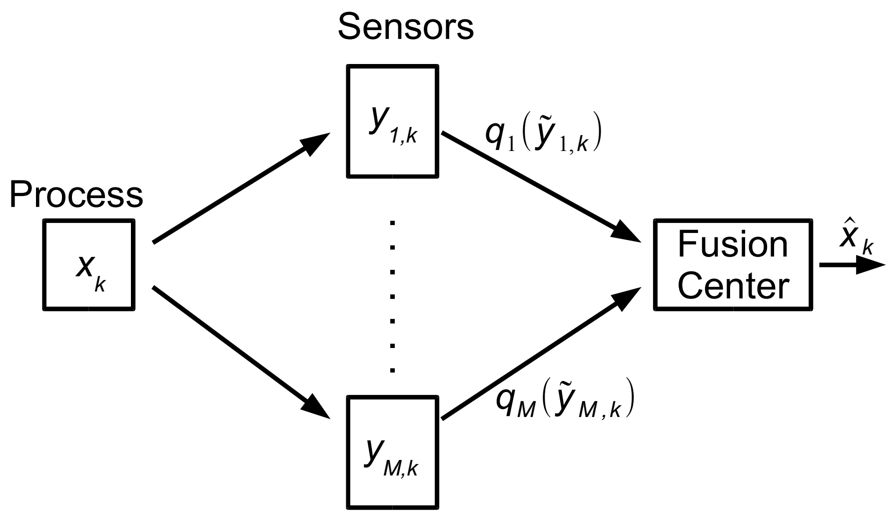
Single Sensor – Asymptotic Analysis

- Therefore

$$P_\infty = P_\infty^{kf} + \delta_N \Phi_1 + \dots$$

where $\delta_N = \begin{cases} \frac{\pi\sqrt{3}}{2N^2} & , \quad \text{optimal quantization} \\ \frac{4\ln N}{3N^2} & , \quad \text{optimal uniform quantization} \end{cases}$

Multiple Sensors



- Vector system $x_{k+1} = Ax_k + w_k$
- *M* sensors with scalar measurements

$$y_{i,k} = C_i x_k + v_{i,k}, \quad i = 1, \ldots, M$$

- Detectability at all sensors assumed (without this, the problem is much harder and currently under investigation)

Multiple Sensors – Decentralized Kalman Filter

- In the case with no quantization, [Hashemipour et al. 1988]
  - Sensors run individual Kalman filters using local information
  - Fusion centre combines local estimates to form global estimate
  - Global estimate same as fusion centre having access to individual sensor measurements

$$\hat{x}^{kf}_{k|k-1} = A\hat{x}^{kf}_{k-1|k-1}$$

$$\hat{x}^{kf}_{k|k} = P^{kf}_{k|k}\Big(P^{kf^{-1}}_{k|k-1}\hat{x}^{kf}_{k|k-1} + \sum_{i=1}^{M}\Big\{P^{kf^{-1}}_{i,k|k}\hat{x}^{kf}_{i,k|k} - P^{kf^{-1}}_{i,k|k-1}\hat{x}^{kf}_{i,k|k-1}\Big\}\Big)$$

$$P^{kf}_{k|k-1} = AP^{kf}_{k-1|k-1}A^T + \Sigma_w$$

$$P^{kf}_{k|k} = P^{kf}_{k|k-1} - P^{kf}_{k|k-1}\mathbf{C}^T(\mathbf{C}P^{kf}_{k|k-1}\mathbf{C}^T + \Sigma_v)^{-1}\mathbf{C}P^{kf}_{k|k-1}$$

- $\hat{x}^{kf}_{i,k|k+1}, \hat{x}^{kf}_{i,k|k}, P^{kf}_{i,k|k+1}, P^{kf}_{i,k|k}$ are local quantities computed at individual sensors
  - Can be reconstructed at fusion centre if sensors send local innovations

Multiple Sensors - Quantized Filtering Scheme

- Modify the scheme of [Hashemipour et al. 1988]
- Individual sensors run:

$$\hat{x}_{i,k|k-1} = A\hat{x}_{i,k-1|k-1}$$

$$\hat{x}_{i,k|k} = \hat{x}_{i,k|k-1} + K_{i,k} l_{i,k} q_{i,k} \left(\frac{y_{i,k} - C_i \hat{x}_{i,k|k-1}}{l_{i,k}} \right)$$

$$K_{i,k} = P_{i,k|k-1} C_i^T (C_i P_{i,k|k-1} C_i^T + \Sigma_{i,v} + \Sigma_{i,n,k})^{-1}$$

$$P_{i,k|k-1} = A P_{i,k-1|k-1} A^T + \Sigma_w$$

$$P_{i,k|k} = P_{i,k|k-1} - P_{i,k|k-1} C_i^T (C_i P_{i,k|k-1} C_i^T + \Sigma_{i,v} + \Sigma_{i,n,k})^{-1} C_i P_{i,k|k-1}$$

- Fusion centre runs:

$$\hat{x}_{k|k-1} = A\hat{x}_{k-1|k-1}$$

$$\hat{x}_{k|k} = P_{k|k}\left(P_{k|k-1}^{-1}\hat{x}_{k|k-1} + \sum_{i=1}^{M}\left\{P_{i,k|k}^{-1}\hat{x}_{i,k|k} - P_{i,k|k-1}^{-1}\hat{x}_{i,k|k-1}\right\}\right)$$

$$P_{k|k-1} = A P_{k-1|k-1} A^T + \Sigma_w$$

$$P_{k|k} = P_{k|k-1} - P_{k|k-1}\mathbf{C}^T(\mathbf{C}P_{k|k-1}\mathbf{C}^T + \Sigma_v + \Sigma_{n,k})^{-1}\mathbf{C}P_{k|k-1}$$

Multiple Sensors - Quantized Filtering Scheme

- Sensor *i* uses either asymptotically optimal uniform quantizer of *Ni* quantization levels or "optimal" quantizer of *Ni* quantization levels

- We have $\Sigma_{i,n,k} = \delta_{i,N_i}(C_i P_{i,k} C_i^T + \Sigma_{i,v})$
  where

$$\delta_{i,N_i} = \begin{cases} \frac{\pi\sqrt{3}}{2N_i^2} & , \quad \text{optimal quantization} \\ \frac{4\ln N_i}{3N_i^2} & , \quad \text{optimal uniform quantization} \end{cases}$$

- *li,k* are updated as in single sensor case
- Provided that *Ni* is sufficiently large that the filter is stable when restricted to any single sensor, then stability of the quantized filtering scheme for multiple sensors will also hold.

Multiple Sensors – Asymptotic Analysis

- Study the behaviour of P_∞ as $N_i \to \infty, \forall i$
- From analysis of single sensor case, we have

$$P_{i,\infty} = P_{i,\infty}^{kf} + O\left(\delta_{i,N_i}\right)$$

- Making use of this result and similar techniques to single sensor case, can find that

$$P_\infty = P_\infty^{kf} + \sum_{i=1}^{M} \delta_{i,N_i} \Phi_{1,i} + \sum_{i,j} O(\delta_{i,N_i} \delta_{j,N_j})$$

where $\Phi_{1,i}$ satisfy Lyapunov equations

$$\Phi_{1,i} = \left(A - A\Phi_0 \mathbf{C}^T (\mathbf{C}\Phi_0\mathbf{C}^T + \Sigma_v)^{-1}\mathbf{C}\right) \Phi_{1,i} \left(A - A\Phi_0 \mathbf{C}^T (\mathbf{C}\Phi_0\mathbf{C}^T + \Sigma_v)^{-1}\mathbf{C}\right)^T$$
$$+ A\Phi_0\mathbf{C}^T(\mathbf{C}\Phi_0\mathbf{C}^T + \Sigma_v)^{-1} F_i (\mathbf{C}\Phi_0\mathbf{C}^T + \Sigma_v)^{-1}\mathbf{C}\Phi_0 A^T$$

Multiple Sensors – Rate Allocation

- Want to allocate a total rate R_{tot} amongst the sensors
- Sensor *i* has rate $R_i = \log_2(N_i)$

- One possible formulation is to minimize trace of asymptotic expression for P_∞ subject to

$$\sum_{i=1}^{M} R_i = R_{tot}$$

- Will obtain discrete optimization problems

Multiple Sensors – Rate Allocation

- For uniform quantization, the discrete optimization problem is

$$\min_{R_1,\ldots,R_M \in \mathbb{Z}^+} \operatorname{tr}(P_\infty^{kf}) + \sum_{i=1}^{M} \frac{e_i R_i}{2^{2R_i}} \text{ s.t. } \sum_{i=1}^{M} R_i = R_{tot}$$

where $e_i = \frac{4 \ln 2}{3} \operatorname{tr}(\Phi_{1,i})$

- If we relax assumption that *Ri* is integer, have the problem

$$\min_{\alpha_1,\ldots,\alpha_M} \operatorname{tr}(P_\infty^{kf}) + \sum_{i=1}^{M} \frac{e_i \alpha_i R_{tot}}{2^{2\alpha_i R_{tot}}}, \text{ s.t. } \sum_{i=1}^{M} \alpha_i = 1, \alpha_i \geq 0$$

- However, this relaxed problem is still non-convex

Multiple Sensors – Rate Allocation

- For optimal quantization, the discrete optimization problem is

$$\min_{R_1,\ldots,R_M \in \mathbb{Z}^+} \mathrm{tr}(P_\infty^{kf}) + \sum_{i=1}^{M} \frac{e_i}{2^{2R_i}} \text{ s.t. } \sum_{i=1}^{M} R_i = R_{tot}$$

where now $e_i = \frac{\pi\sqrt{3}}{2}\mathrm{tr}(\Phi_{1,i})$

- If we relax assumption that *Ri* is integer, have the problem

$$\min_{\alpha_1,\ldots,\alpha_M} \mathrm{tr}(P_\infty^{kf}) + \sum_{i=1}^{M} \frac{e_i}{2^{2\alpha_i R_{tot}}}, \text{ s.t. } \sum_{i=1}^{M} \alpha_i = 1, \alpha_i \geq 0$$

- Lemma: The optimal solution to relaxed problem is

$$\alpha_i^* = \frac{1}{M} + \frac{1}{2R_{tot}} \log_2 \frac{e_i}{\left(\prod_{j=1}^{M} e_j\right)^{1/M}}$$

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Numerical Studies

- System parameters:

$$A = \begin{bmatrix} 1.2 & 0.5 \\ 0 & 1.1 \end{bmatrix}, \quad \Sigma_w = I$$

- Single sensor case:

$$C = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad \Sigma_v = 1$$

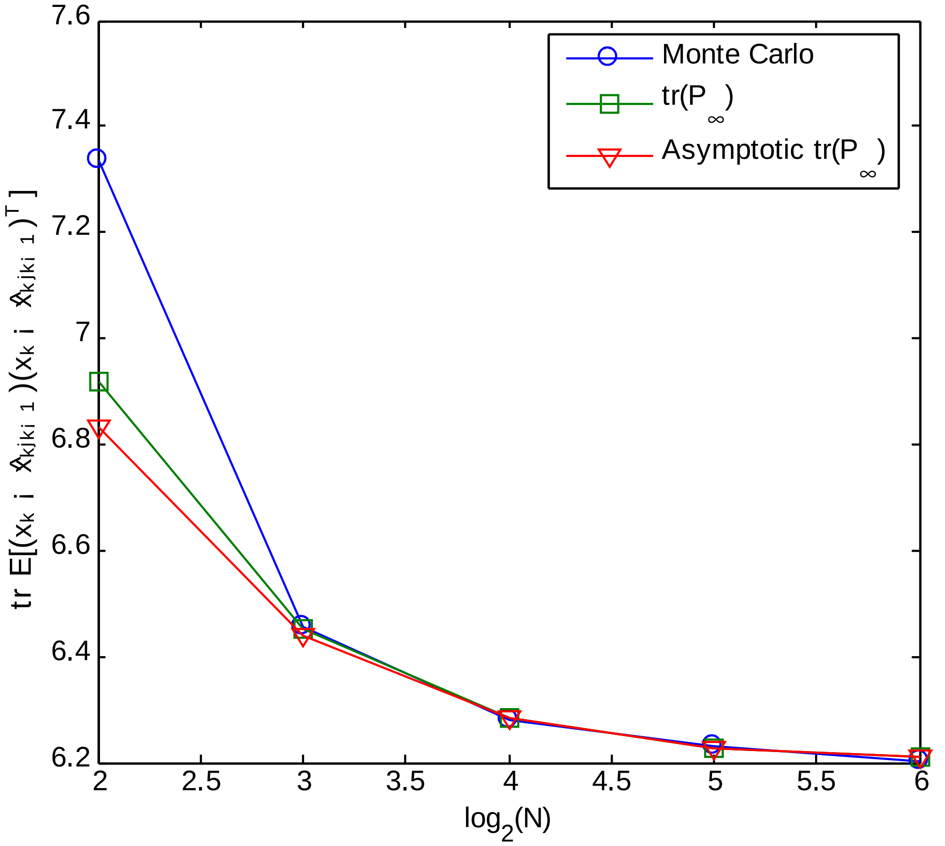
- Two sensors case:

$$C_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad \Sigma_{1,v} = 1$$

$$C_2 = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad \Sigma_{2,v} = 0.2$$

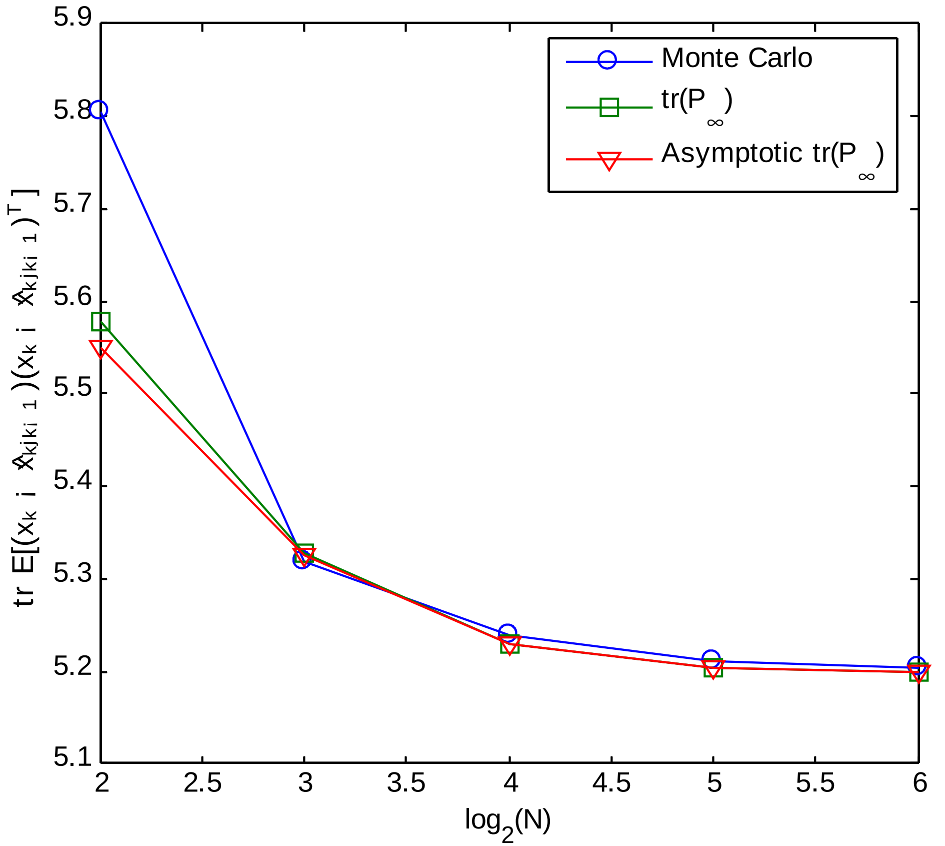
Numerical Studies

- Single sensor, uniform quantizer



Numerical Studies

- Two sensors, optimal quantizer, *N1=N2=N*



Numerical Studies

- Two sensors, uniform quantization
- Rate allocation, $R_{tot} = 8$

R_1	R_2	Monte Carlo	$\text{tr}(P_\infty)$	Asymptotic $\text{tr}(P_\infty)$
2	6	5.242	5.2181	5.2232
3	5	5.237	5.2177	5.2179
4	4	5.247	5.2408	5.2405
5	3	5.322	5.3166	5.3212
6	2	5.585	5.4886	5.5271

Numerical Studies

- Two sensors, optimal quantization
- Rate allocation, $R_{tot} = 8$

R_1	R_2	Monte Carlo	$\mathrm{tr}(P_\infty)$	Asymptotic $\mathrm{tr}(P_\infty)$
2	6	5.474	5.2213	5.2321
3	5	5.219	5.2119	5.2124
4	4	5.240	5.2290	5.2289
5	3	5.315	5.3136	5.3185
6	2	6.306	5.5996	5.6829

- Solving the relaxed problem gives $\alpha_1^* = 0.3798, \alpha_2^* = 0.6202$, corresponding to rates $R_1^* = 3.0386, R_2^* = 4.9614$

Conclusions and further work

- Derived asymptotic expression relating estimation error with quantization rates of sensors
- Sketched a proof of stability of the scheme
- Considered a rate allocation problem

- Further areas of investigation
  - Packet loss and high rate quantization
  - Vector measurements: dynamic quantization for lattice vector quantizers
  - Detectability at all sensors a strong assumption
  - Low data rates?
    - Proof of stability here holds for sufficiently high bit rates
    - May need different schemes to achieve stability for lower bit rates
    - Tradeoff between estimation performance and data rate for rates close to minimum bit rates