Optimal Collaborative Control in the Absence of Communication

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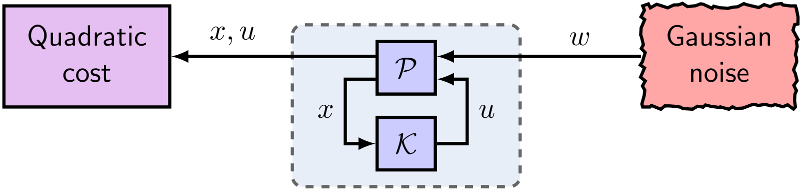
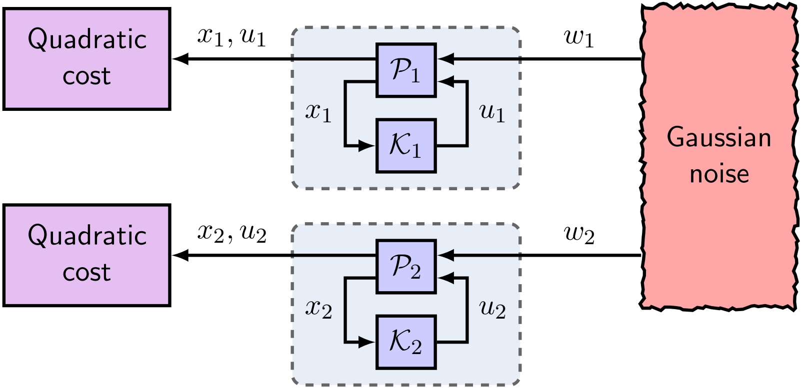


- $\mathcal{P}$ is an LTI system with state x
- w is i.i.d. Gaussian noise
- Infinite-horizon cost

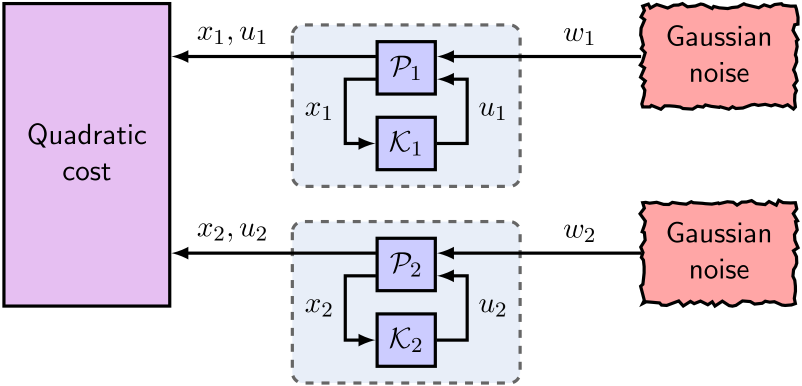
Optimal Policy

static feedback: $u = Kx$



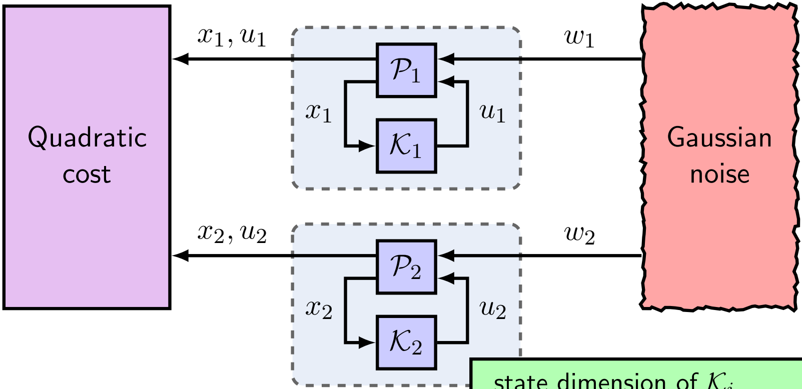
Optimal Policy

greedy static feedback: $u_i = K_i x_i$



Optimal Policy

greedy static feedback: $u_i = K_i x_i$



state dimension of $\mathcal{K}_i$

$= (n_1 + n_2)\,\mathbf{rank}\,\mathbf{E}(w_1 w_2^\mathsf{T})$

Optimal Policy

dynamic feedback:
$$\dot{q}_i = A_{Ki} q_i + B_{Ki} x_i$$
$$u_i = C_{Ki} q_i + D_{Ki} x_i$$

Problem statement

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} e$$

$$z = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} D_1 & D_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

where e is white noise. Find a control policy that minimizes:

$$\mathcal{J} = \left(\lim_{T \to \infty} \frac{1}{T} \, \mathbf{E} \int_0^T \left\| z(t) \right\|^2 \mathrm{d}t \right)^{1/2}$$

- u_1 only measures x_1
- u_2 only measures x_2

Problem statement

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} e$$

$$z = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} D_1 & D_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

- **Linear controllers:** $\begin{aligned} \dot{q}_i &= A_{Ki} q_i + B_{Ki} x_i \\ u_i &= C_{Ki} q_i + D_{Ki} x_i \end{aligned}$ for $i = 1, 2$
- **Stability:** A_1, A_2 are Hurwitz
- **Normalization:** $NN^\mathsf{T} = \begin{bmatrix} I & X \\ X^\mathsf{T} & I \end{bmatrix}$
- **Rank assumption:** $\begin{bmatrix} A - j\omega I & B \\ C & D \end{bmatrix}$ and D; full column rank

Coordinate transformation

LQR formulation

$$\dot{x} = Ax + Bu + Ne$$
$$z = Cx + Du$$

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \mathcal{K}_1 & 0 \\ 0 & \mathcal{K}_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

minimize $\mathcal{J}$

subject to $\mathcal{K}_i \in \mathcal{RL}_\infty$

Model-matching formulation

$$\begin{bmatrix} \mathcal{F} & \mathcal{G} \end{bmatrix} = \left[\begin{array}{c|cc} A & N & B \\ \hline C & 0 & D \end{array}\right]$$

minimize $\left\| \mathcal{F} + \mathcal{G} \begin{bmatrix} \mathcal{Q}_1 & 0 \\ 0 & \mathcal{Q}_2 \end{bmatrix} N \right\|_2$

subject to $\mathcal{Q}_i \in \mathcal{RH}_2$

$$\mathcal{Q} = \mathcal{K}\left(I - \mathcal{T}B\mathcal{K}\right)^{-1}\mathcal{T}$$

where: $\mathcal{T} = (sI - A)^{-1}$

Overview of approach

1. Solve the model-matching problem

$$\begin{array}{ll} \text{minimize} & \left\| \mathcal{F} + \mathcal{G} \begin{bmatrix} \mathcal{Q}_1 & 0 \\ 0 & \mathcal{Q}_2 \end{bmatrix} N \right\|_2 \\ \text{subject to} & \mathcal{Q}_1, \mathcal{Q}_2 \in \mathcal{RH}_2 \end{array}$$

2. Lemma: If $\mathcal{Q}_i^{\mathsf{opt}} = \left[\begin{array}{c|c} \hat{A}_i & \hat{B}_i \\ \hline \hat{C}_i & 0 \end{array}\right]$, the optimal controller is

$$\mathcal{K}_i^{\mathsf{opt}} = \left[\begin{array}{c|c} \hat{A}_i - \hat{B}_i B_i \hat{C}_i & (\hat{A}_i - \hat{B}_i B_i \hat{C}_i)\hat{B}_i - \hat{B}_i A_i \\ \hline \hat{C}_i & \hat{C}_i \hat{B}_i \end{array}\right]$$

3. Check for minimality

A simpler problem

$$\begin{array}{ll} \text{minimize} & \|\mathcal{F} + \mathcal{G}\mathcal{Q}N\|_2 \\ \text{subject to} & \mathcal{Q} \in \mathcal{RH}_2 \end{array}$$

Solution: If $\begin{bmatrix} \mathcal{F} & \mathcal{G} \end{bmatrix} = \left[\begin{array}{c|cc} A & N & B \\ \hline C & 0 & D \end{array}\right]$ then

$$\mathcal{Q}^{\text{opt}} = \left[\begin{array}{c|c} A + BK & I \\ \hline K & 0 \end{array}\right] \quad \text{and} \quad \mathcal{K}^{\text{opt}} = K$$

where $K = \mathbf{care}(A, B, C, D)$.

Optimality condition: $\mathcal{G}^*(\mathcal{F} + \mathcal{G}\mathcal{Q}N)N^{\mathsf{T}} \in \mathcal{H}_2^{\perp}$

Optimality conditions

minimize	$\left\| \mathcal{F} + \mathcal{G} \begin{bmatrix} \mathcal{Q}_1 & 0 \\ 0 & \mathcal{Q}_2 \end{bmatrix} N \right\|_2$
subject to	$\mathcal{Q}_1, \mathcal{Q}_2 \in \mathcal{RH}_2$

Optimality condition:

$$\mathcal{G}^* \left(\mathcal{F} + \mathcal{G} \begin{bmatrix} \mathcal{Q}_1 & 0 \\ 0 & \mathcal{Q}_2 \end{bmatrix} N \right) N^\mathsf{T} \in \begin{bmatrix} \mathcal{H}_2^\perp & \mathcal{L}_2 \\ \mathcal{L}_2 & \mathcal{H}_2^\perp \end{bmatrix}$$

rewrite as:

$$\mathcal{G}^* \mathcal{F} N^\mathsf{T} + \mathcal{G}^* \mathcal{G} \begin{bmatrix} \mathcal{Q}_1 & 0 \\ 0 & \mathcal{Q}_2 \end{bmatrix} \begin{bmatrix} I & X \\ X^\mathsf{T} & I \end{bmatrix} \in \begin{bmatrix} \mathcal{H}_2^\perp & \mathcal{L}_2 \\ \mathcal{L}_2 & \mathcal{H}_2^\perp \end{bmatrix}$$

Decomposition

$$\mathcal{G}^*\mathcal{F}N^\mathsf{T} + \mathcal{G}^*\mathcal{G}\begin{bmatrix}\mathcal{Q}_1 & 0\\ 0 & \mathcal{Q}_2\end{bmatrix}\begin{bmatrix} I & X\\ X^\mathsf{T} & I\end{bmatrix} \in \begin{bmatrix}\mathcal{H}_2^\perp & \mathcal{L}_2\\ \mathcal{L}_2 & \mathcal{H}_2^\perp\end{bmatrix}$$

Extract diagonal blocks...

$$\left[\mathcal{G}^*\mathcal{F}N^\mathsf{T}\right]_{11} + E_1^\mathsf{T}\mathcal{G}^*\mathcal{G}\begin{bmatrix}\mathcal{Q}_1\\ \mathcal{Q}_2 X^\mathsf{T}\end{bmatrix} \in \mathcal{H}_2^\perp$$

$$\left[\mathcal{G}^*\mathcal{F}N^\mathsf{T}\right]_{22} + E_2^\mathsf{T}\mathcal{G}^*\mathcal{G}\begin{bmatrix}\mathcal{Q}_1 X\\ \mathcal{Q}_2\end{bmatrix} \in \mathcal{H}_2^\perp$$

Take the SVD: $X = \begin{bmatrix}U_1 & \bar{U}_1\end{bmatrix}\begin{bmatrix}\Sigma & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}U_2 & \bar{U}_2\end{bmatrix}^\mathsf{T}$, where $\Sigma < I$.

Key idea: The coupled equations in $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are decoupled in $\mathcal{Q}_1 U_1$, $\mathcal{Q}_1 \bar{U}_1$, $\mathcal{Q}_2 U_2$, and $\mathcal{Q}_2 \bar{U}_2$.

Decoupled equations

Multiplying by $\bar{U}_1$ and $\bar{U}_2$...

$$\begin{aligned}\left[\mathcal{G}^*\mathcal{F}N^\mathsf{T}\right]_{11}\bar{U}_1 + \left[\mathcal{G}^*\mathcal{G}\right]_{11}\left(\mathcal{Q}_1\bar{U}_1\right) \in \mathcal{H}_2^\perp \\ \left[\mathcal{G}^*\mathcal{F}N^\mathsf{T}\right]_{22}\bar{U}_2 + \left[\mathcal{G}^*\mathcal{G}\right]_{22}\left(\mathcal{Q}_2\bar{U}_2\right) \in \mathcal{H}_2^\perp\end{aligned}$$

Multiplying by U_1 and U_2...

$$\left[\mathcal{G}^*\mathcal{F}N^\mathsf{T}\right]_{11}U_1 + E_1^\mathsf{T}\mathcal{G}^*\mathcal{G}\begin{bmatrix}\mathcal{Q}_1U_1 \\ \mathcal{Q}_2U_2\Sigma\end{bmatrix} \in \mathcal{H}_2^\perp$$

$$\left[\mathcal{G}^*\mathcal{F}N^\mathsf{T}\right]_{22}U_2 + E_2^\mathsf{T}\mathcal{G}^*\mathcal{G}\begin{bmatrix}\mathcal{Q}_1U_1\Sigma \\ \mathcal{Q}_2U_2\end{bmatrix} \in \mathcal{H}_2^\perp$$

Extract k^{th} column ($k = 1, \dots, r$):

$$\begin{bmatrix}\left[\mathcal{G}^*\mathcal{F}N^\mathsf{T}\right]_{11}U_1 \\ \left[\mathcal{G}^*\mathcal{F}N^\mathsf{T}\right]_{22}U_2\end{bmatrix} e_k + \begin{bmatrix}[\mathcal{G}^*\mathcal{G}]_{11} & \sigma_k[\mathcal{G}^*\mathcal{G}]_{12} \\ \sigma_k[\mathcal{G}^*\mathcal{G}]_{21} & [\mathcal{G}^*\mathcal{G}]_{22}\end{bmatrix}\begin{bmatrix}\mathcal{Q}_1U_1e_k \\ \mathcal{Q}_2U_2e_k\end{bmatrix} \in \mathcal{H}_2^\perp$$

Solution procedure

Optimality conditions correspond to optimization problems. e.g.

$$\left[\mathcal{G}^*\mathcal{F}N^{\mathsf{T}}\right]_{11}\bar{U}_1 + \left[\mathcal{G}^*\mathcal{G}\right]_{11}\left(\mathcal{Q}_1\bar{U}_1\right) \in \mathcal{H}_2^{\perp}$$

corresponds to:

$$\begin{array}{rl} \text{minimize} & \left\|\mathcal{F}N^{\mathsf{T}}E_1\bar{U}_1 + \mathcal{G}E_1\left(\mathcal{Q}_1\bar{U}_1\right)\right\|_2 \\ \text{subject to} & \left(\mathcal{Q}_1\bar{U}_1\right) \in \mathcal{RH}_2 \end{array}$$

and its solution is:

$$\mathcal{Q}_1\bar{U}_1 = \left[\begin{array}{c|c} A_1 + B_1K_1 & \bar{U}_1 \\ \hline K_1 & 0 \end{array}\right]$$

where $K_1 = \mathbf{care}(A_1, B_1, C_1, D_1)$.

Final solution

1. Solve $K_i = \mathbf{care}(A_i, B_i, C_i, D_i)$ for $i = 1, 2$.
2. Solve $\hat{K}_k = \mathbf{care}\left(A, B, \begin{bmatrix} \sqrt{\frac{\sigma_k}{1-\sigma_k}} C \\ C_1 E_1^\mathsf{T} \\ C_2 E_2^\mathsf{T} \end{bmatrix}, \begin{bmatrix} \sqrt{\frac{\sigma_k}{1-\sigma_k}} D \\ D_1 E_1^\mathsf{T} \\ D_2 E_2^\mathsf{T} \end{bmatrix}\right)$ for $k = 1 : r$
3. Define the quantities:

$$A_L = \begin{bmatrix} A + B\hat{K}_1 & & \\ & \ddots & \\ & & A + B\hat{K}_r \end{bmatrix} \quad B_{Li} = \begin{bmatrix} \hat{N} e_1 e_1^\mathsf{T} U_i^\mathsf{T} \\ \vdots \\ \hat{N} e_r e_r^\mathsf{T} U_i^\mathsf{T} \end{bmatrix} \quad \hat{N} = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

$$C_{Ki} = \begin{bmatrix} E_i^\mathsf{T} \hat{K}_1 - K_i E_i^\mathsf{T} & \cdots & E_i^\mathsf{T} \hat{K}_r - \end{bmatrix}$$

$$D_{Ki} = K_i \bar{U}_i \bar{U}_i^\mathsf{T} + E_i^\mathsf{T} \sum_{k=1}^{r} \hat{K}_k \hat{N} e_k e_k^\mathsf{T} U_i^\mathsf{T}$$

state dimension of $\mathcal{K}_i$

$$= \begin{cases} rn - n_i & \text{if } r = n_i \\ rn & \text{otherwise} \end{cases}$$

$$\mathcal{K}_i = \left[\begin{array}{c|c} A_L - B_{Li} B_i C_{Ki} & A_L B_{Li} - B_{Li}(A_i + B_i D_{Ki}) \\ \hline C_{Ki} & D_{Ki} \end{array}\right]$$

Optimal decentralized cost

$$\mathcal{J}_{\text{opt}}^2 = \left\| \left[\begin{array}{c|c} A_1 + B_1 K_1 & I \\ \hline C_1 + D_1 K_1 & 0 \end{array}\right] \bar{U}_1 \right\|^2 + \left\| \left[\begin{array}{c|c} A_2 + B_2 K_2 & I \\ \hline C_2 + D_2 K_2 & 0 \end{array}\right] \bar{U}_2 \right\|^2 + 2 \sum_{k=1}^{r} \left\| \left[\begin{array}{c|c} A + B\hat{K}_k & I \\ \hline \hat{C}_k + \hat{D}_k \hat{K}_k & 0 \end{array}\right] \begin{bmatrix} \frac{1}{\sqrt{2}} U_1 e_k \\ \frac{1}{\sqrt{2}} U_2 e_k \end{bmatrix} \right\|^2$$

Orthogonal columns

Optimal cost assuming noise is uncorrelated (greedy policy)

Optimal centralized cost using:

$$\begin{bmatrix} x_1 \\ u_1 \\ x_2 \\ u_2 \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} Q_{11} & S_{11} & \sigma_k Q_{12} & \sigma_k S_{12} \\ S_{11}^{\mathsf{T}} & R_{11} & \sigma_k S_{21}^{\mathsf{T}} & \sigma_k R_{12} \\ \sigma_k Q_{21} & \sigma_k S_{21} & Q_{22} & S_{22} \\ \sigma_k S_{12}^{\mathsf{T}} & \sigma_k R_{21} & S_{22}^{\mathsf{T}} & R_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ u_1 \\ x_2 \\ u_2 \end{bmatrix}$$

Optimal decentralized cost

$$X = \begin{bmatrix} \underbrace{U_1}_{r} & \underbrace{\bar{U}_1}_{n_1 - r} \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \underbrace{U_2}_{r} & \underbrace{\bar{U}_2}_{n_2 - r} \end{bmatrix}^\mathsf{T}$$

$$\begin{aligned}
\mathcal{J}_{\text{opt}}^2 = &\left\| \left[\begin{array}{c|c} A_1 + B_1 K_1 & I \\ \hline C_1 + D_1 K_1 & 0 \end{array} \right] \bar{U}_1 \right\|^2 \\
&+ \left\| \left[\begin{array}{c|c} A_2 + B_2 K_2 & I \\ \hline C_2 + D_2 K_2 & 0 \end{array} \right] \bar{U}_2 \right\|^2 \\
&+ 2 \sum_{k=1}^{r} \left\| \left[\begin{array}{c|c} A + B\hat{K}_k & I \\ \hline \hat{C}_k + \hat{D}_k \hat{K}_k & 0 \end{array} \right] \begin{bmatrix} \frac{1}{\sqrt{2}} U_1 e_k \\ \frac{1}{\sqrt{2}} U_2 e_k \end{bmatrix} \right\|^2
\end{aligned}$$

Greedy upper bound:

$$\mathcal{J}_{\text{opt}}^2 \leq \left\| \left[\begin{array}{c|c} A_1 + B_1 K_1 & I \\ \hline C_1 + D_1 K_1 & 0 \end{array} \right] \right\|^2 + \left\| \left[\begin{array}{c|c} A_2 + B_2 K_2 & I \\ \hline C_2 + D_2 K_2 & 0 \end{array} \right] \right\|^2$$

- Achieved when $r = 0$ or when all $\sigma_k \to 0$.
- (greedy strategy is optimal)

Optimal decentralized cost

$$X = \begin{bmatrix} \underbrace{U_1}_{r} & \underbrace{\bar{U}_1}_{n_1-r} \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \underbrace{U_2}_{r} & \underbrace{\bar{U}_2}_{n_2-r} \end{bmatrix}^{\mathsf{T}}$$

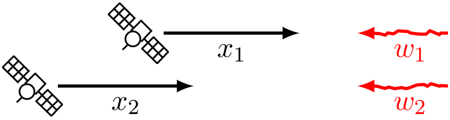
$$\begin{aligned} \mathcal{J}_{\text{opt}}^2 = &\left\| \left[\begin{array}{c|c} A_1 + B_1 K_1 & I \\ \hline C_1 + D_1 K_1 & 0 \end{array} \right] \bar{U}_1 \right\|^2 \\ &+ \left\| \left[\begin{array}{c|c} A_2 + B_2 K_2 & I \\ \hline C_2 + D_2 K_2 & 0 \end{array} \right] \bar{U}_2 \right\|^2 \\ &+ 2 \sum_{k=1}^{r} \left\| \left[\begin{array}{c|c} A + B\hat{K}_k & I \\ \hline \hat{C}_k + \hat{D}_k \hat{K}_k & 0 \end{array} \right] \begin{bmatrix} \frac{1}{\sqrt{2}} U_1 e_k \\ \frac{1}{\sqrt{2}} U_2 e_k \end{bmatrix} \right\|^2 \end{aligned}$$

Centralized lower bound:

$$\left\| \left[\begin{array}{c|c} A + BK & N \\ \hline C + DK & 0 \end{array} \right] \right\|^2 \leq \mathcal{J}_{\text{opt}}^2$$

- Achieved when $r = n_1 = n_2$ and all $\sigma_k \to 1$.
- (decentralized and centralized have same performance)

Numerical example



Two satellites with velocities x_1 and x_2 respectively.

$$\dot{x}_1 = -2x_1 + u_1 + w_1$$
$$\dot{x}_2 = -3x_2 + u_2 + w_2$$

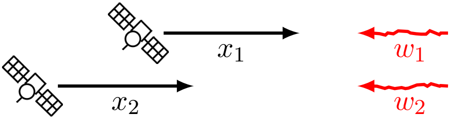
Objective: match velocities while keeping effort low.

Cost parameter: $z^2 = (x_1 - x_2)^2 + 0.01(u_1^2 + u_2^2)$.

Noise covariance: $\mathbf{E}(w_1 w_2) = 0.8$.

Numerical example

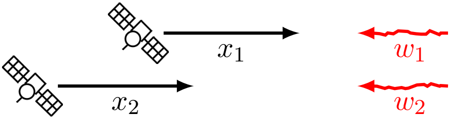
$\mathbf{E}(w_1 w_2) = 0.8$



Policy	Control Law	Cost
No control	$u_1 = u_2 = 0$	0.3109
Greedy	$u_1 = -8.1980 x_1$ $u_2 = -7.4403 x_2$	0.2808
Optimal	$\dot{q}_1 = -13.79 q_1 - 0.2659 x_1$ $u_1 = 4.255 q_1 - 3.013 x_1$ $\dot{q}_2 = -13.52 q_2 + 0.2659 x_2$ $u_2 = 4.255 q_2 - 2.279 x_2$	0.2300
Centralized	$u_1 = -6.3095 x_1 + 5.9121 x_2$ $u_2 = 5.9121 x_1 - 5.6051 x_2$	0.1567

Numerical example

$\mathbf{E}(w_1 w_2) = 0.9$



Policy	Control Law	Cost
No control	$u_1 = u_2 = 0$	0.2380
Greedy	$u_1 = -8.1980 x_1$ $u_2 = -7.4403 x_2$	0.2630
Optimal	$\dot{q}_1 = -14.16 q_1 - 0.2766 x_1$ $u_1 = 4.979 q_1 - 1.921 x_1$ $\dot{q}_2 = -13.88 q_2 + 0.2766 x_2$ $u_2 = 4.979 q_2 - 1.198 x_2$	0.1766
Centralized	$u_1 = -6.3095 x_1 + 5.9121 x_2$ $u_2 = 5.9121 x_1 - 5.6051 x_2$	0.1128

Further observations

- $\mathcal{K}_{\text{greedy}}$ is *not* the best static decentralized controller!
  - $\mathcal{K}_{\text{greedy}}$ may perform worse than $\mathcal{K} = 0$.
  - Sometimes $\mathcal{K}_{\text{opt}}$ is static, yet $\mathcal{K}_{\text{opt}} \neq \mathcal{K}_{\text{greedy}}$.
- $\mathcal{K}_{\text{opt}}$ may be unstable, even for a stable plant!

Open questions

For this simple problem

- How do we interpret the controller states?
  What is the sufficient statistic?

- Is there an information theory interpretation?
  Is information measured in dimensions? (Sahai)

- What is the *common information*?
  Is there a fictitious coordinator? (Teneketzis)

Possible extensions

- More than two subsystems (difficult).

- Output feedback (difficult).

Thank you!

This work will appear in Proc. Allerton Conference, 2012
Paper available at: www.laurentlessard.com