Disturbance Propagation in Leader-Follower Systems

FOCUS

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joint work with

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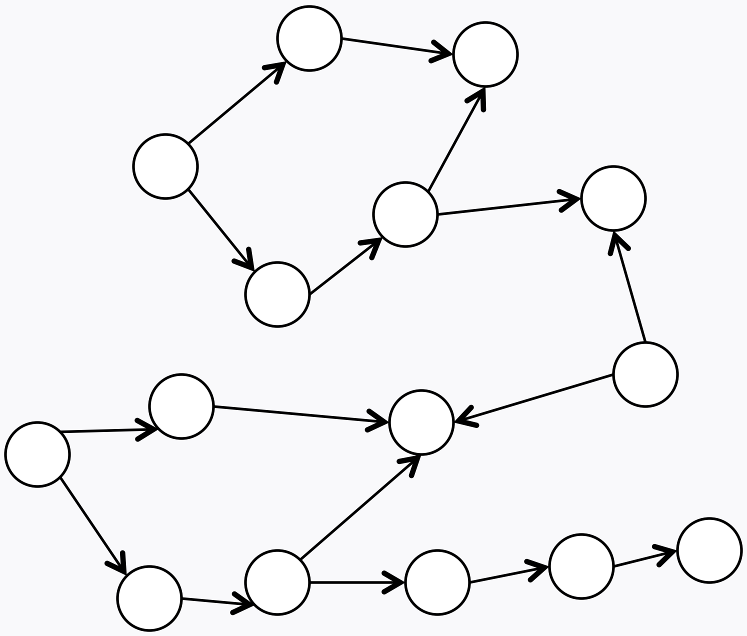
Formation Control Problem

- In general, it is a hard problem
- How to design controllers?
- How to design the information graph?
- How do we choose the leaders?



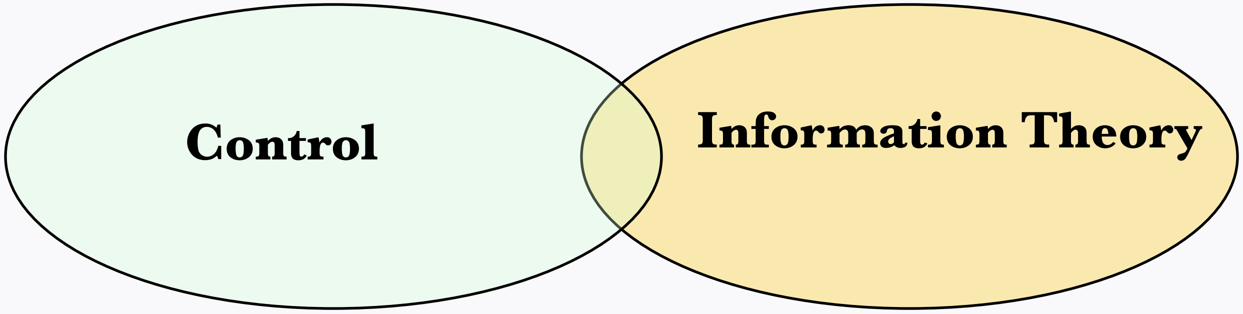
- In this talk, we focus on performance limitations

Results in a Nutshell

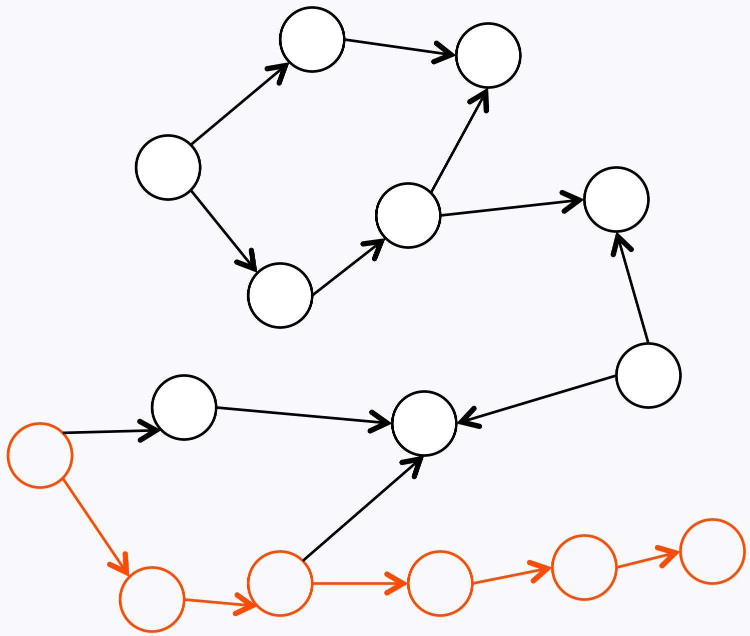
- We focus on the sensitivity of the agents' position with respect to an external disturbance

- Generalize Bode integral formula for SISO systems to distributed systems
  - Fundamental limitation that holds for any plant

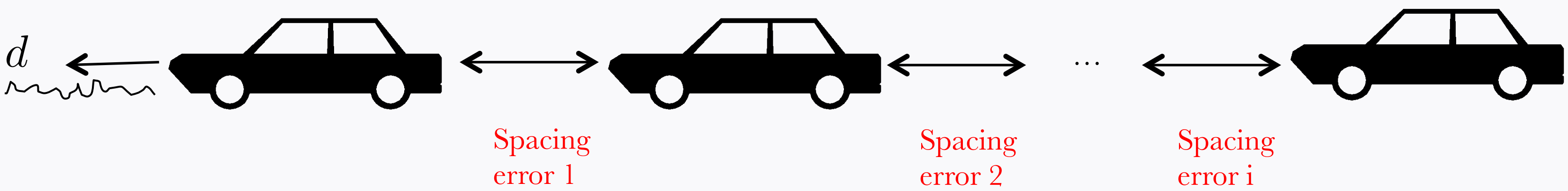
- Focus on the stochastic setting and make use of information-theoretic tools



Car Platoon Systems



Automated Highway Systems



Related Literature

- Stability analysis
  - Chu (1974), Peppard (1974), Swaroop and Hedrick (1996)

Predecessor following strategy

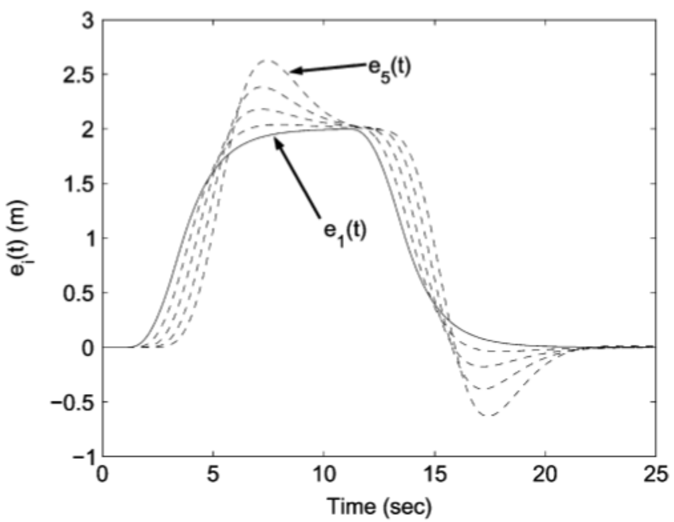


Fig. 5. Time domain plots of spacing errors with the predecessor following strategy.

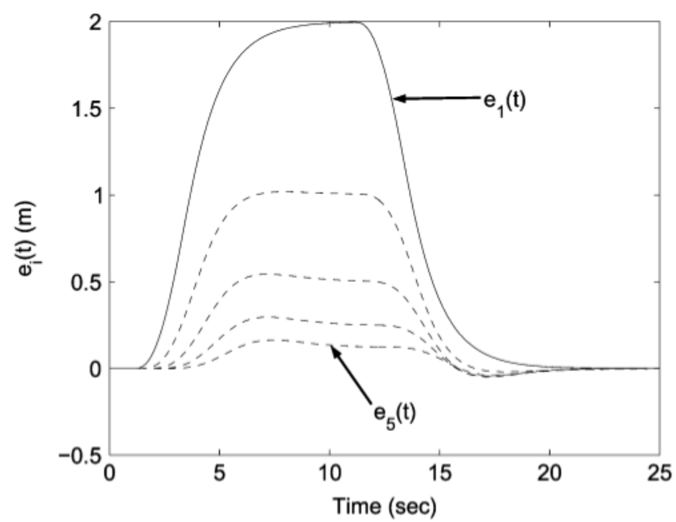


Fig. 6. Time domain plots of spacing errors with the predecessor and leader following strategy.

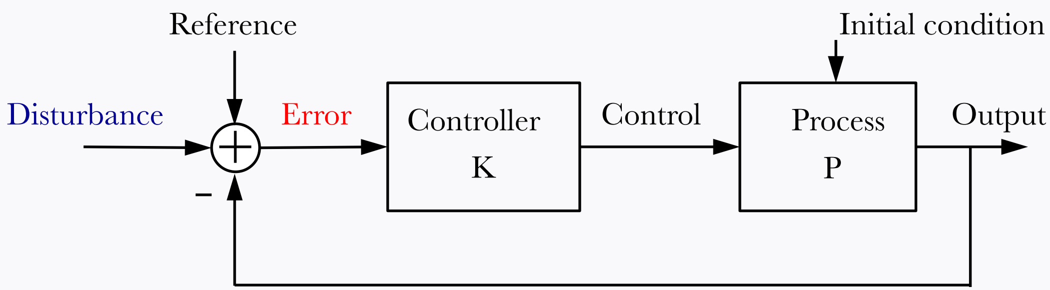
Predecessor and leader following strategy

- Disturbance propagation performance
  - Seiler, Pant, and Hedrick (2004), Middleton and Braslavsky (2010)
- These previous works focus on specific plants and controllers. We provide fundamental performance results that hold for any plant

Outline

- Bode Integral formulae for SISO plants
  - Deterministic
  - Stochastic
- Generalization to platoon systems under predecessor following strategy
  - Deterministic
  - Stochastic
- Extensions to the leader and predecessor following strategy
- Concluding remarks

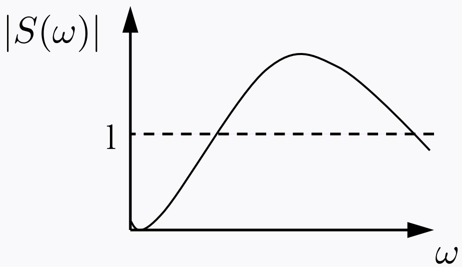
Bode Integral Formula: Sensitivity



- Sensitivity function (from disturbance to error):

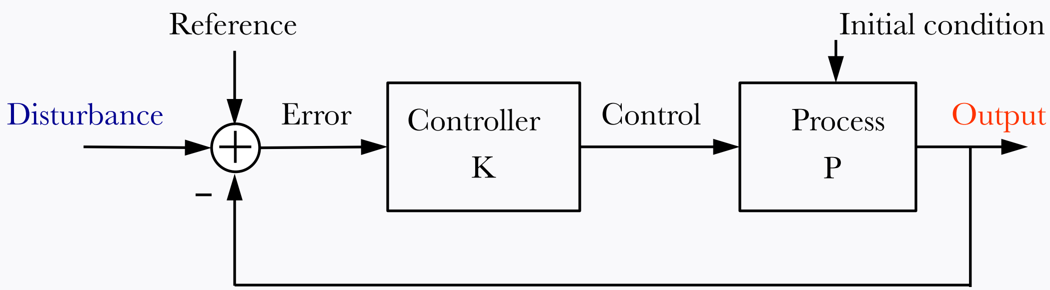
$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|S(\omega)|d\omega = \sum_{\lambda \in \mathcal{U}} \log|\lambda|$$

Unstable poles



- This limitation holds for any LTI control
- Application of Jensens' formula in complex analysis
- Extensions of Bode formula for LTI systems
  - Freudenberg and Looze (1985), Freudenberg and Looze (1988)
  - Mohtadi (1990), Chen (1995)

Bode Integral Formula: Complementary Sensitivity



- Complementary sensitivity function (from disturbance to output):

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|T(\omega)|d\omega = \sum_{\beta\in\mathcal{Z}} \log|\beta| + \sum_{\beta'\in\mathcal{Z}_K} \log|\beta'| + \log|GD|$$

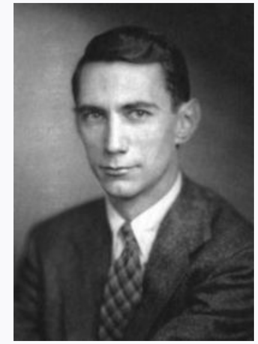
Unstable plant/controller zeros

Plant/controller gain

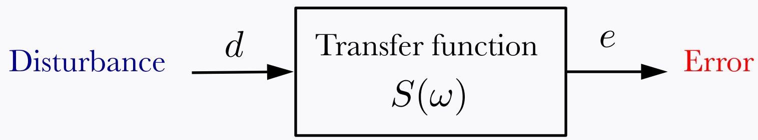
- Controller plays a role now

- If K and P are minimum phase, then the limitation is only given by the loop gain

Bode Integral Formula and Information Theory



- Szego's limit theorems for Toeplitz matrices:



1) Stochastic disturbance through a linear stable filter with transfer function S

$$\bar{h}(e) - \bar{h}(d) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |S(\omega)| d\omega$$

2) If d and e are WSS process with power spectral densities $P_d(\omega)$ and $P_e(\omega)$

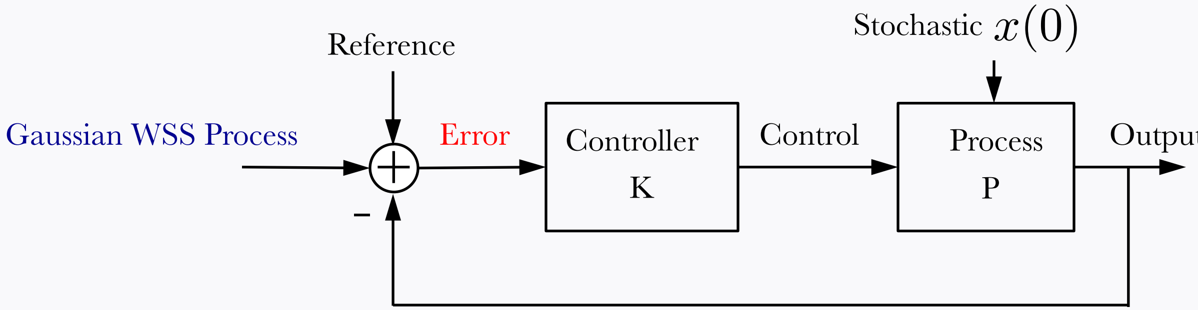
$$P_e(\omega) = |S(\omega)|^2 P_d(\omega) \quad \Longrightarrow \quad S(\omega) = \sqrt{\frac{P_e(\omega)}{P_d(\omega)}} =: \mathcal{S}_{ed}(\omega)$$

$$\bar{h}(x) \leq \frac{1}{4\pi} \int_{-\pi}^{\pi} \log 2\pi e P(\omega) d\omega$$

Related Literature

- Connections between Bode Integral formula and information theory
  - Iglesias (2001): Nonlinear control
  - Zhang and Iglesias (2003): Nonlinear control
  - Elia (2004): Stabilization and Gaussian feedback capacity
  - Martins, Dahleh, and Doyle (2007): Bode formula with disturbance preview
  - Martins and Dahleh (2008): Stochastic Bode formula
  - Okano, Hara, and Ishii (2009): Complementary sensitivity
  - Ishii, Okano, and Hara (2011): Stochastic Bode formula MIMO case
  - Yu and Mehta (2010): Nonlinear control
  - Ardestanizadeh and Franceschetti (2012): Gaussian channels with memory

Stochastic Bode Integral Formula: Sensitivity



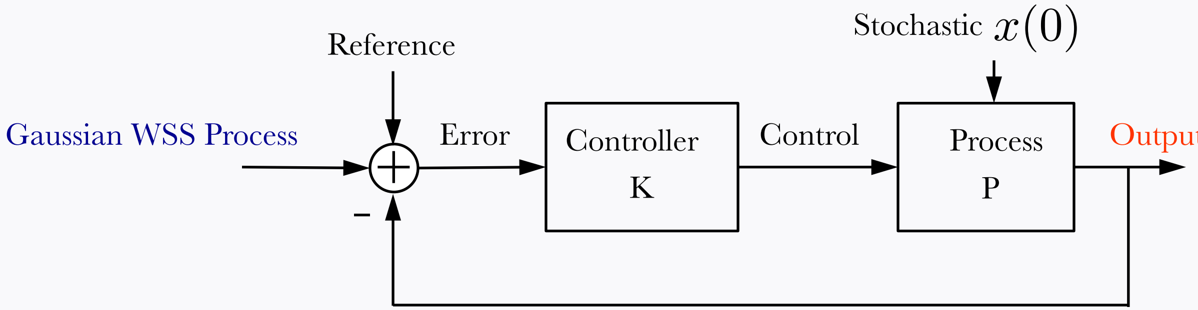
- Martins and Dahleh (2008):

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log \mathcal{S}(\omega) d\omega \geq \sum_{\lambda \in \mathcal{U}} \log|\lambda|$$

- This limitation holds for any 2^{nd} moment stabilizing control (including nonlinear)
- The disturbance and x(0) are independent

$$\begin{aligned} \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|\mathcal{S}(\omega)| &\geq \bar{h}(e) - \bar{h}(d) \\ &\geq \liminf_{k\to\infty} \tfrac{1}{k} I(x(0); e^k) \\ &\geq \sum_{\lambda \in \mathcal{U}} \log|\lambda| \end{aligned}$$

Stochastic Bode Formula: Complementary Sensitivity



- Okano, Hara, and Ishii (2009)

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|\mathcal{T}(\omega)|d\omega \geq \sum_{\beta\in\mathcal{Z}}\log|\beta| + \log|GD|$$

Unstable plant zeros Plant/controller gain

- This limitation holds for any 2nd moment stabilizing LTI control
- The disturbance and x(0) are independent
- K's zeros are not present because the initial condition is assumed deterministic
- If P is minimum phase or if x(0) is deterministic then the limitation is only given by the loop gain

Stochastic Bode Formula: Complementary Sensitivity

- Proof based on bounds on the entropy rates

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log\mathcal{T}(\omega)d\omega = \frac{1}{4\pi}\int_{-\pi}^{\pi}\log 2\pi e P_y(\omega)d\omega - \frac{1}{4\pi}\int_{-\pi}^{\pi}\log 2\pi e P_d(\omega)d\omega$$

$$\geq \bar{h}(y) - \bar{h}(d)$$

$$\geq \liminf_{k\to\infty}\frac{1}{k}I(x(0);y^k) + \log|GD|$$

$$\geq \sum_{\beta\in\mathcal{Z}}\log|\beta| + \log|GD|$$

- The unstable zeros are the poles of the inverse systems, which are related to the eigenvalues of the system matrix

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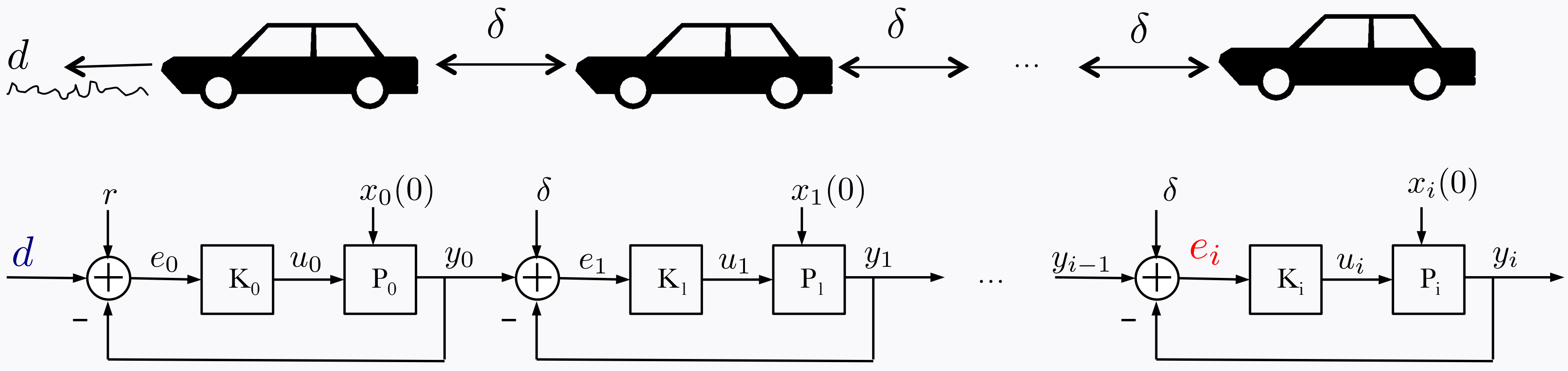
Leader-Follower Platoon Control: Problem Setup

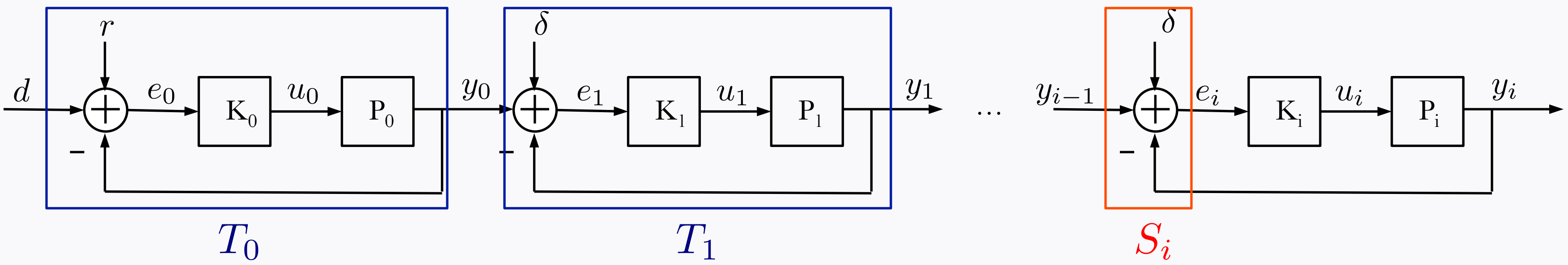


- Disturbance d is a WSS Gaussian process
- d is independent of the initial conditions
- The initial conditions form a Markov sequence $x_0(0) \to x_1(0) \to \ldots \to x_i(0)$
- Closed loop systems are stable and steady state analysis (all processes are WSS)
- Sensitivity of the i-th spacing error e_i to the stochastic disturbance

$$\mathcal{S}_i := \sqrt{\frac{P_{e_i}(\omega)}{P_d(\omega)}}$$

Platoon System: Deterministic Setting

- The transfer function from d to e_i factorizes as



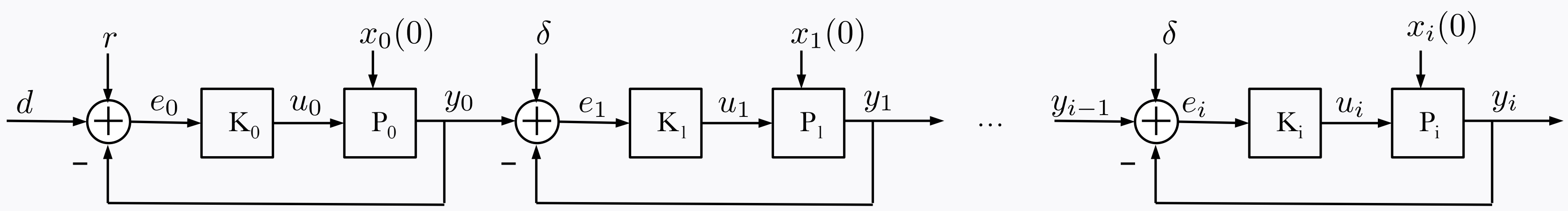
- Hence, combining the Bode integral formulae for deterministic SISO systems

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|S_{de_i}(\omega)|d\omega = \sum_{l=0}^{i-1}\left(\sum_{\beta\in Z_K\cup Z}\log\beta + \log(G_iD_i)\right) + \sum_{\lambda\in\mathcal{U}_i}\log|\lambda|$$

Unstable zeros   Loop gain   Unstable poles

- Holds for any stable LTI controller at the i-th follower

Platoon System: Stochastic Setting



- We could follow a similar modular approach

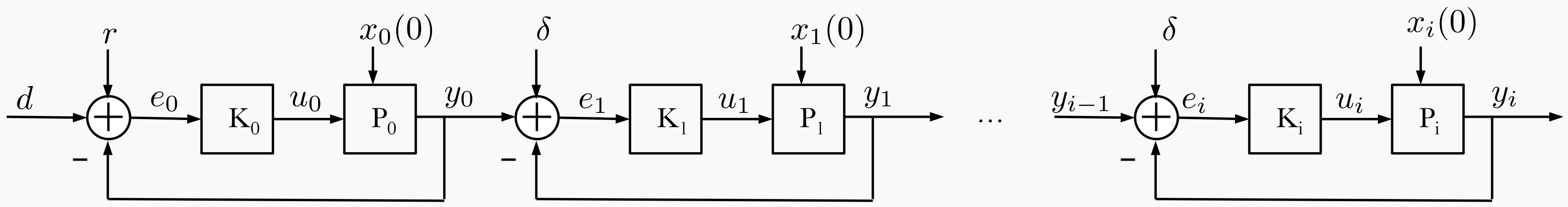
$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log \mathcal{S}_i(\omega)d\omega \geq \bar{h}(e_i) - \bar{h}(d)$$

$$= \bar{h}(e_i) - \bar{h}(y_{i-1}) + \bar{h}(y_{i-1}) + \cdots + \bar{h}(y_0) - \bar{h}(d)$$

- And then apply the results by Martins and Dahleh (2008) and Okano, Hara, and Ishii (2009)

- However, these results require independence between the disturbance and the plant initial condition and the result on the complementary sensitivity requires LTI controllers.

Main Result



- If the controllers are LTI:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log \mathcal{S}_i(\omega) d\omega \geq \sum_{l=0}^{i-1} \log(G_l D_l) + \sum_{\lambda \in \mathcal{U}_i} \log|\lambda|$$

Loop gain   Unstable poles

- No unstable zeros at the predecessors' controller/plant:
  1. The controller initial conditions are deterministic
  2. The plant initial conditions are correlated: In the worst case scenario they are fully correlated and deterministically known

Remarks

- Consistent with deterministic case if all closed-loop systems are minimum phase
- It can be tight in non-trivial cases, e.g., when all processes are jointly Gaussian for some suitably chosen linear controllers
- It can be extended to the case where the controllers are nonlinear (but differentiable and one-to-one):

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log \mathcal{S}_i(\omega) d\omega \geq \sum_{l=0}^{i-1}\left(\log G_l + \boxed{U_l}\right) + \sum_{\lambda \in \mathcal{U}_i} \log |\lambda|$$

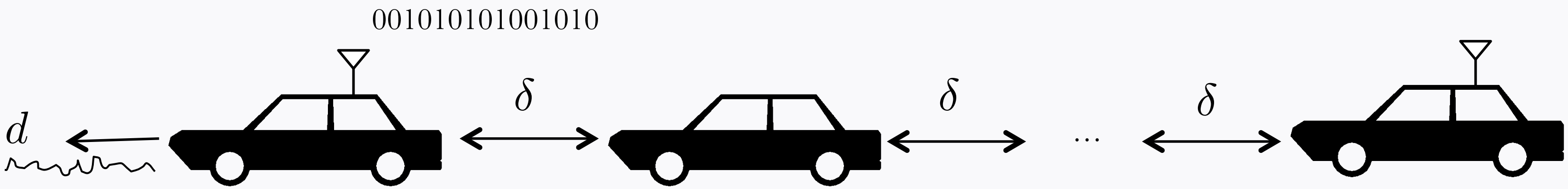
$$U_i := \liminf_{k\to\infty} \frac{1}{k}\sum_{i=0}^{k} \mathbb{E}\left(\log |u'(e_k)|\right)$$

- Consequence of the scaling property of differential entropy:

$$h(\phi(x)) = \mathbb{E}(\phi'(x)) + h(x)$$

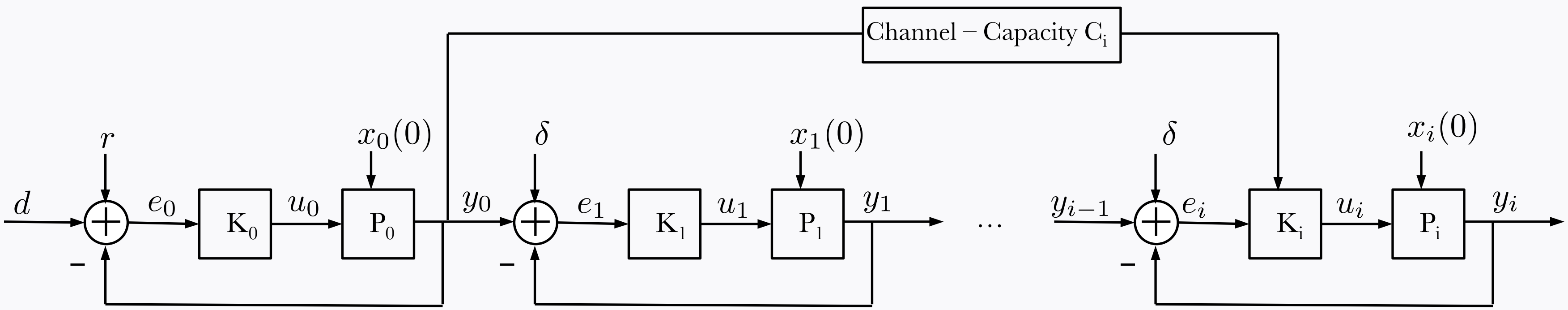
Leader-Predecessor Following Strategy

- Suppose that the leader can send information to each follower over finite capacity channels



Communication Channels

- The leader channel output is communicated to the i-th follower, i=2,3,…, over a communication channel of finite Shannon capacity C_i



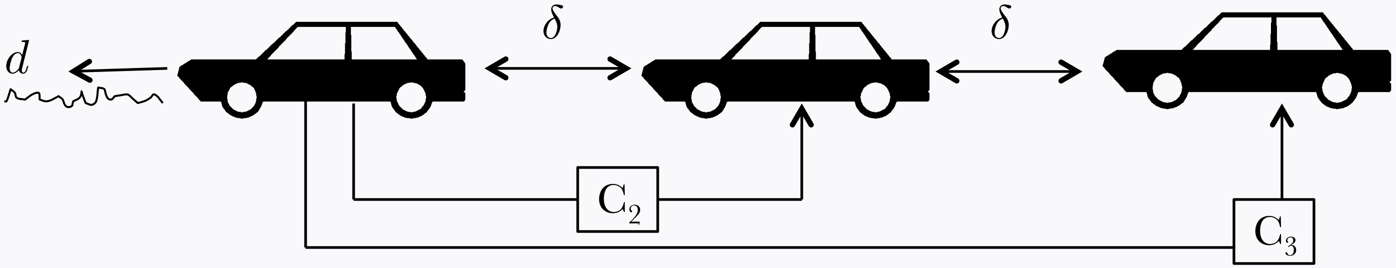
- If the controllers are LTI:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log \mathcal{S}_i(\omega)d\omega \geq \sum_{l=0}^{i-1}\left(\log(G_l D_l) - C_l\right)^+ + \sum_{\lambda\in\mathcal{U}_i}\log|\lambda| - C_i$$

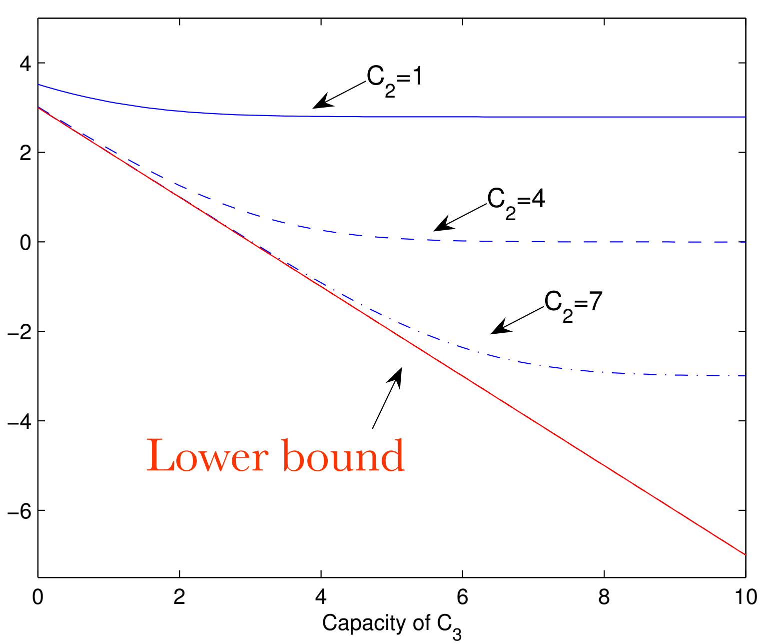
- The right hand side reduces thanks to the disturbance preview
- There is a saturation effect: The reduction is no greater than the loop gain

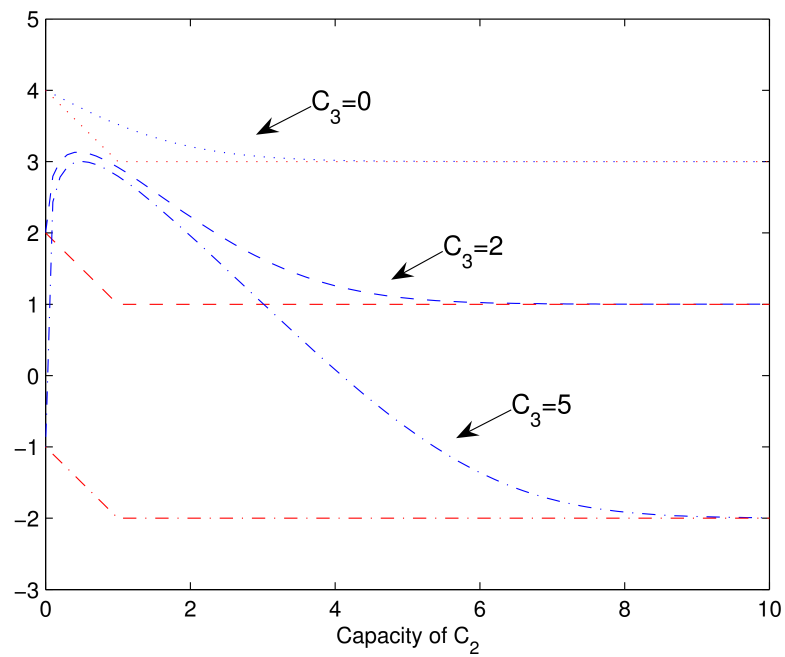
Numerical Example

Specific Gaussian setting where the sensitivity can be evaluated analytically



Lower bound vs Integral log sensitivity





Concluding Remarks

- Two approaches have been followed to study performance limitations

	Deterministic	Stochastic
Approach	Frequency domain	Time domain
Tools	Complex Analysis	Information theory
Assumptions	1. Transfer function must exist 2. LTI controllers	1. WSS processes 2. 2^{nd} moment stable plants 3. Stochastic initial conditions 4. Disturbance and initial conditions are independent

- We have followed the stochastic approach to provide performance bounds in one-dimensional formation control problems
- Immediate extension to trees
- Currently working on graphs with loops
- Communication graph vs sensing graph