Closed-Loop Stabilization Over a Gaussian Interference Channel

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Outline

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   - Sufficient Condition for Stability
   - Numerical Results
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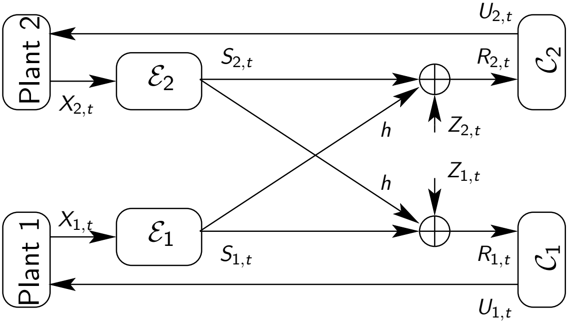
Closed-loop System

Two discrete-time LTI systems (plants)

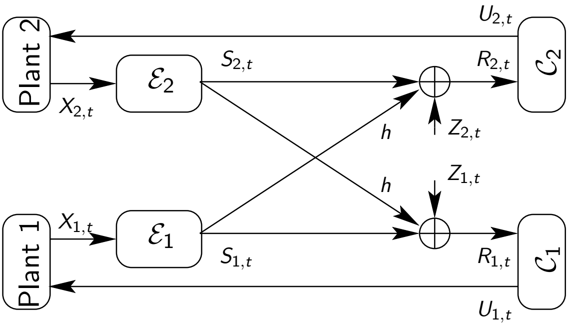
$$X_{i,t+1} = \lambda_i X_{i,t} + U_{i,t} + W_{i,t}, \qquad t \geq 0, \quad i = 1,2$$

- The initial states $X_{i,0}$, have arbitrary pdfs with finite variance.
- Variance of $X_{i,t}$ is denoted by $\alpha_{i,t}$.
- Cross correlation coefficient, $\rho_t := \mathbb{E}[X_{1,t}X_{2,t}]/\sqrt{\alpha_{1,t}\alpha_{2,t}}$.
- Process noise, $W_{i,t} \sim \mathcal{N}(0, n_w)$.
- The open loop systems are unstable, $\lambda_i > 1$.

Control Over Interference Channel



- $Z_{1,t} \sim \mathcal{N}(0, N)$ and $Z_{2,t} \sim \mathcal{N}(0, N)$ are white noise components.
- Noise cross-correlation coefficient : $\rho_z \triangleq \frac{\mathbb{E}[Z_{1,t}Z_{2,t}]}{N} \in [-1, 1]$
- Cross channel gain, $h \in \mathbb{R}_+$.



- The i-th sensor policy $f_{i,t}$:

$$S_{i,t} = f_{i,t}(X_{i,[0,t]}), \qquad \text{with } \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}[S_{i,t}^2] \leq P,$$

- The i-th controller receives, $R_{i,t} = S_{i,t} + hS_{j,t} + Z_{i,t}$.
- The i-th controller policy, $U_{i,t} = \pi_{i,t}(R_{i,[0,t]})$.

Closed-loop System

Objective: Find conditions on $\{\lambda_1, \lambda_2\}$ such that the *remote controllers* can *mean square stabilize* the two systems over the given Interference Channel.

Definition: Mean Square Stability

A system is said to be *mean square stable* if there exists a constant $M < \infty$ such that $\mathbb{E}[||X_t||^2] < M$ for all t.

Related Works: On Control Over Gaussian channels

R. Bansal and T. Başar
Simultaneous Design of Measurement and Control Strategies for Stochastic Systems with Feedback. *Automatica*,1989.

S. Tatikonda, A. Sahai, and S. Mitter
Stochastic Linear Control Over a Communication Channel. *IEEE Trans. on Automatic Control*, 2004.

Braslavsky, R. Middleton, and J. Freudenberg,
Feedback Stabilization Over Signal-to-Noise Ratio Constrained Channels . *IEEE Trans. Automat. Control*, 2008.

S. Yüksel and T. Başar
Control Over Noisy Forward and Reverse Channels. *IEEE Trans. Automat. Control*, 2011.

G. Lipsa and N. C. Martins
Optimal Memoryless Control of a Delay in Gaussian Noise: A Simple Counterexample. *Automatica*, 2011.

On Coding Schemes for Gaussian Channels

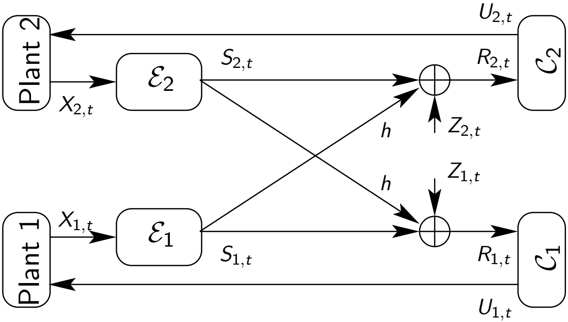
N. Elia
When Bode meets Shannon: controloriented feedback communication schemes. *IEEE Trans. Autom. Control*, 2004.

Schalkwijk and Kailath
A coding scheme for additive noise channels with feedback–I: no bandwidth constraint. *IEEE Trans. Inform. Theory*, 1966.

Gerhard Kramer
Feedback strategies for white Gaussian interference networks. *IEEE Trans. Inform. Theory*, 2002.

M. Gastpar, A. Lapidoth, and M. Wigger
When feedback doubles the prelog in AWGN networks. *IEEE Trans. Inform. Theory*, submitted.

Necessary Condition for Stabilization



Theorem

The two LTI systems can be mean square stabilized over the given symmetric Gaussian interference channel only if

$$\log\left(|\lambda_i|\right) \leq \frac{1}{2}\log\left(1 + \frac{P\left(1+h\right)^2}{N}\right), \quad i = 1, 2.$$

Proof:Necessary Condition

Lemma

The i-th linear system can be mean square stabilized over the Gaussian interference channel only if

$$\log\left(|\lambda_i|\right) \leq \liminf_{T\to\infty} \frac{1}{T} I\left(S_{i,[0,T-1]} \to R_{i,[0,T-1]}\right), \quad i = 1, 2.$$

- Directed information:
  $I\left(S_{i,[0,T-1]} \to R_{i,[0,T-1]}\right) = \sum_{t=0}^{T-1} I\left(S_{i,[0,t]}; R_{i,t} | R_{i,[0,t\text{-}1]}\right).$

S. Yüksel
Characterization of information channels for aymptotic mean stationarity and stochastic stability of non-stationary linear systems. *IEEE Trans. Inform. Theory*,To appear.

Zaidi, Oechtering, Yüksel, and Skoglund
Stabilization of linear systems over Gaussian relay networks. *IEEE Trans. Autom. Control*, Submitted June 2012.

Proof:Necessary Condition

$$
\begin{aligned}
& I\left(S_{i,[0,T-1]} \to R_{i,[0,T-1]}\right) \overset{(a)}{\leq} I\left(S_{i,[0,T-1]}; R_{i,[0,T-1]}\right) \\
& \overset{(b)}{\leq} I\left(S_{1,[0,T-1]}, S_{2,[0,T-1]}; R_{i,[0,T-1]}\right) \\
& \overset{(c)}{=} h\left(R_{i,[0,T-1]}\right) - h\left(R_{i,[0,T-1]} | S_{1,[0,T-1]}, S_{2,[0,T-1]}\right) \\
& = h\left(R_{i,[0,T-1]}\right) - h\left(Z_{i,[0,T-1]}\right) \overset{(d)}{=} \sum_{t=0}^{T-1} \left[h\left(R_{i,t} | R_{i,[0,t-1]}\right) - h\left(Z_{i,t}\right)\right] \\
& \overset{(e)}{\leq} \sum_{t=0}^{T-1} \left[h\left(R_{i,t}\right) - h\left(Z_{i,t}\right)\right] \\
& \overset{(f)}{=} \sum_{t=0}^{T-1} \frac{1}{2} \log\left(1 + \frac{P\left(1 + h^2 + 2h\rho_t\right)}{N}\right) \overset{(g)}{\leq} \frac{T}{2} \log\left(1 + \frac{P\left(1+h\right)^2}{N}\right).
\end{aligned}
$$

Proof:Necessary Condition

We have shown that:

$$I\left(S_{i,[0,T-1]} \to R_{i,[0,T-1]}\right) \leq \frac{T}{2} \log\left(1 + \frac{P\left(1+h\right)^2}{N}\right) \tag{1}$$

Lemma

The i-th linear system can be mean square stabilized over the Gaussian interference channel only if

$$\log\left(|\lambda_i|\right) \leq \liminf_{T\to\infty} \frac{1}{T} I\left(S_{i,[0,T-1]} \to R_{i,[0,T-1]}\right), \tag{2}$$

- Using (1) in (2), we get:

$$\log\left(|\lambda_i|\right) \leq \frac{1}{2} \log\left(1 + \frac{P\left(1+h\right)^2}{N}\right), \quad i = 1, 2. \quad \square$$

Sufficient Condition for Stability

Theorem (Sufficient condition)

The two LTI systems can be mean square stabilized over the given white Gaussian interference channel if

$$\log(\lambda_i) < \frac{1}{2}\log\left(\frac{P\left(1+h^2+2h\rho^\star\right)+N}{Ph^2\left(1-\rho^{\star 2}\right)+N}\right),$$

where $\rho^\star$ is the largest among all the roots in the interval $[0,1]$ of the following two fourth order polynomials

$$f_1(\rho) := \rho^4 + a_3\rho^3 + a_2\rho^2 + a_1\rho + a_0,$$
$$f_2(\rho) := \rho^4 + b_3\rho^3 + b_2\rho^2 + b_1\rho + b_0,$$

where $\{a_i, b_i\}$ are given on the next slide.

Main Result–stability result

Theorem (Sufficient condition) – Continued

$$a_3 = \frac{N}{2hP}, \quad a_2 = -2 - \frac{N(4 + h\rho_z)}{2h^2 P},$$
$$a_1 = -\frac{N(1 + 2h^2 + 2h\rho_z)}{2h^3 P} - \frac{N^2}{h^3 P^2},$$
$$a_0 = 1 + \frac{N(2h - \rho_z)}{2h^3 P}, \quad b_3 = \frac{2h^2 P + 2P + N}{2hP},$$
$$b_2 = \frac{N\rho_z}{2hP}, \quad b_1 = -\frac{(1 + h^2)}{h} - \frac{N(1 + 2\rho_z - 2h^2)}{2h^3 P},$$
$$b_0 = -1 - \frac{N(2h - \rho_z)}{2h^3 P}.$$

- Different polynomials for noisy and noiseless systems.

Proof: Initialization

Initial time step, $t = 0$:

- $\mathcal{E}_1$ transmits $S_{1,0} = \sqrt{\frac{P}{\alpha_{1,0}}} X_{1,0}$.
- $\mathcal{E}_2$ doesn't transmit, i.e., $S_{2,0} = 0$.
- $\mathcal{C}_1$ receives $R_{1,0} = S_{1,0} + Z_{1,0}$, and estimates

$$\hat{X}_{1,0} = \sqrt{\frac{\alpha_{1,0}}{P}} R_{1,0} = X_{1,0} + \sqrt{\frac{\alpha_{1,0}}{P}} Z_{1,0}$$

- $\mathcal{C}_1$ takes an action $U_{1,0} = -\lambda_1 \hat{X}_{1,0}$

$$X_{1,1} = \lambda_1 (X_{1,0} - \hat{X}_{1,0}) + W_{1,0} = \sqrt{\frac{\alpha_{1,0}}{P}} Z_{1,0} + W_{1,0}.$$

$\Rightarrow$ New **plant state $X_{1,1}$ is zero mean Gaussian** distributed.

- $\mathcal{C}_2$ doesn't take any action, i.e., $U_{2,0} = 0$.

Proof: Initialization

Initial time step, $t = 1$:

- $\mathcal{E}_1$ doesn't transmit, i.e., $S_{1,0} = 0$.
- $\mathcal{E}_2$ transmits $S_{2,1} = \sqrt{\frac{P}{\alpha_{2,1}}} X_{2,1}$.
- $\mathcal{C}_2$ receives $R_{2,1} = S_{2,1} + Z_{2,1}$, and estimates

$$\hat{X}_{2,1} = \sqrt{\frac{\alpha_{2,1}}{P}} R_{2,1} = X_{2,1} + \sqrt{\frac{\alpha_{2,1}}{P}} Z_{2,1}$$

- $\mathcal{C}_1$ doesn't take any action, i.e., $U_{1,1} = 0$.
- $\mathcal{C}_2$ takes an action $U_{2,1} = -\lambda_2 \hat{X}_{2,1}$

$$X_{2,2} = \lambda_2 (X_{2,1} - \hat{X}_{2,1}) + W_{2,1} = \sqrt{\frac{\alpha_{2,1}}{P}} Z_{2,1} + W_{2,1}.$$

$\Rightarrow$ New **plant state $X_{2,2}$ is zero mean Gaussian** distributed.

Proof: Further transmissions

Further time steps $t \geq 2$:

- $\mathcal{E}_1$ transmits $S_{1,t} = \sqrt{\frac{P}{\alpha_{1,t}}} X_{1,t}$.
- $\mathcal{E}_2$ transmits $S_{2,t} = \sqrt{\frac{P}{\alpha_{2,t}}} X_{2,t} \text{sgn}(\rho_t)$.
- $\mathcal{C}_i$ receives $R_{i,t} = S_{i,t} + hS_{j,t} + Z_{i,t}$, and estimates the

$$\hat{X}_{i,t} = \mathbb{E}[X_{i,t}|R_t] = \frac{\mathbb{E}[R_{i,t}X_{i,t}]}{\mathbb{E}[R_{i,t}^2]} R_{i,t},$$

- $\mathcal{C}_i$ takes an action: $U_{i,t} = -\lambda_i \hat{X}_{i,t}$ for the plant i.
- The new states are: $X_{i,t+1} = \lambda_i (X_{i,t} - \hat{X}_{i,t}) + W_{i,t}$.

Proof: Second Moment of State Process

The second moment of the i-th state process is given by

$$\alpha_{i,t+1} = \alpha_{i,t}\lambda_i^2 \left(\frac{Ph^2\left(1-|\rho_t|^2\right)+N}{P(1+h^2+2h|\rho_t|)+N} \right) + n_w,$$

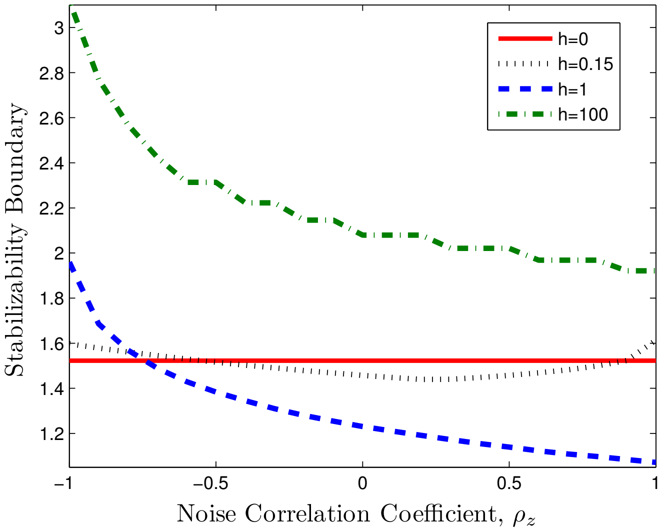
where

$$\rho_{t+1} = \text{sgn}(\rho_t)\lambda_1\lambda_2\sqrt{\frac{\alpha_{1,t}\alpha_{2,t}}{\alpha_{1,t+1}\alpha_{2,t+1}}}\Bigg(-2\frac{P\left(h+|\rho_t|\right)\left(1+|\rho_t|\right)}{P(1+h^2+2h|\rho_t|)+N} + |\rho_t| + \frac{P\left(1+h|\rho_t|\right)^2\left(2hP+P|\rho_t|(1+h^2)+N\rho_z\right)}{\left(P(1+h^2+2h|\rho_t|)+N\right)^2}\Bigg)$$

We show that $\alpha_{i,t} \to M$ as $t \to \infty$ if

$$\log(\lambda_i) < \frac{1}{2}\log\left(\frac{P\left(1+h^2+2h\rho^\star\right)+N}{Ph^2\left(1-\rho^{\star 2}\right)+N}\right) \square$$

Effect of Interference and Noise Correlation



Effect of Interference and Noise Correlation

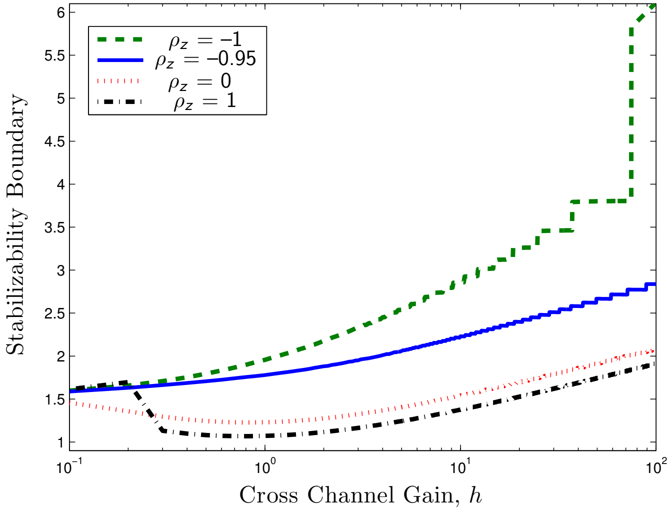
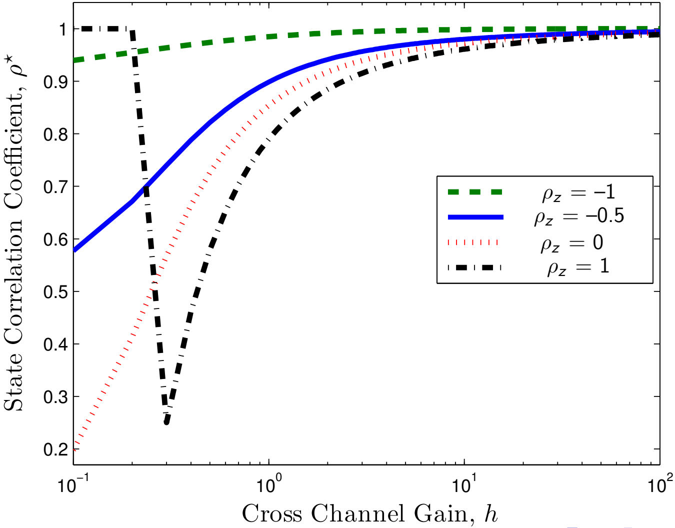


Illustration of cross correlation of the state processes



On optimality of linear policies

- Necessary Condition:

$$\log(\lambda_i) \leq \frac{1}{2}\log\left(1+\frac{P(1+h)^2}{N}\right), \quad i=1,2.$$

- Sufficient Condition:

$$\log(\lambda_i) < \frac{1}{2}\log\left(\frac{P\left(1+h^2+2h\rho^\star\right)+N}{Ph^2\left(1-\rho^{\star 2}\right)+N}\right), \quad i=1,2.$$

- Linear policies are optimal if $\rho^\star = 1$.

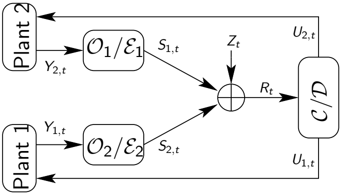
On optimality of linear policies

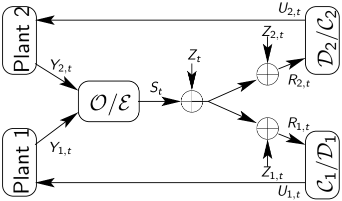
Linear policies are optimal for the following special cases:

1. $\rho_z = 1$ and $2h(1 + \frac{h^2 P}{N}) < 1$.
2. $\rho_z = \pm 1$ and $\rho_0 = 1$.

- There might be other cases where linear scheme is optimal.

Other Multi-user Setups





A. Zaidi, T. Oechtering, and M. Skoglund.
Sufficient Conditions for Closed-Loop Control Over Multiple-Access and Broadcast Channels
IEEE Conf. on Decision and Control (CDC), December 2010.

Concluding Remarks

Problem: Mean square stabilizing two discrete time LTI systems over a white Gaussian interference channel.

- Necessary condition using information theoretic tools.
- Sufficient conditions using delay-free linear time varying scheme (noisy and noiseless cases).
- Behavior under the proposed scheme:
  - Stabilizability improves as the interference gets very strong.
  - Highly correlated state processes in strong interference.
  - Negative noise correlation helps in general.
  - For $\rho_z = -1$, there is a boost in stabilizability.
- Special cases where linear is optimal. Other cases?

Thank you for your attention!