Scalable Design Rules for Physical Systems

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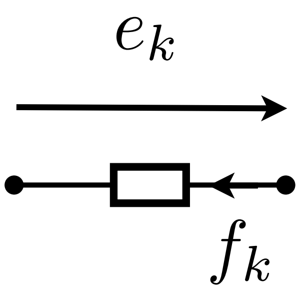
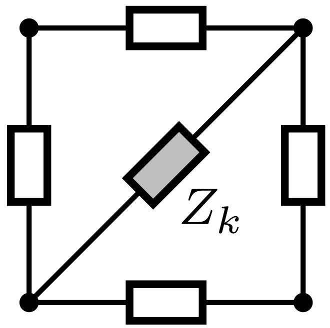
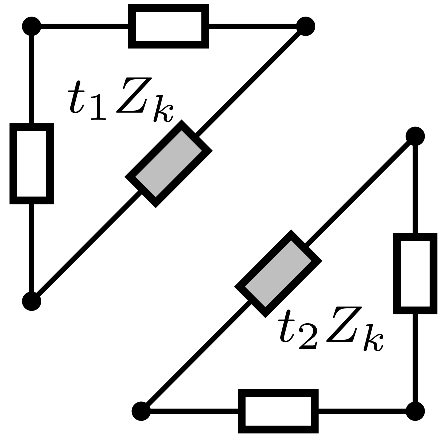
$$e_k = Z_k(s) f_k$$

$$\text{or } e_k = \phi_k(f_k)$$

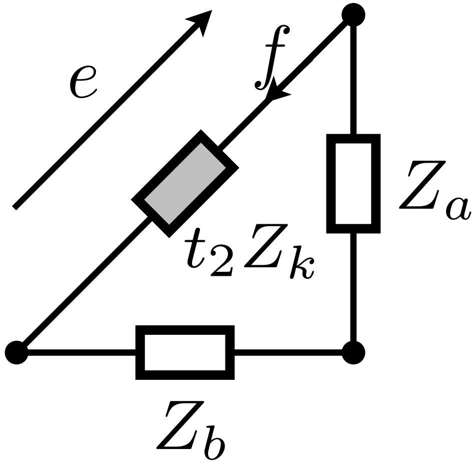


Component





$$\frac{1}{t_1} + \frac{1}{t_2} = 1$$

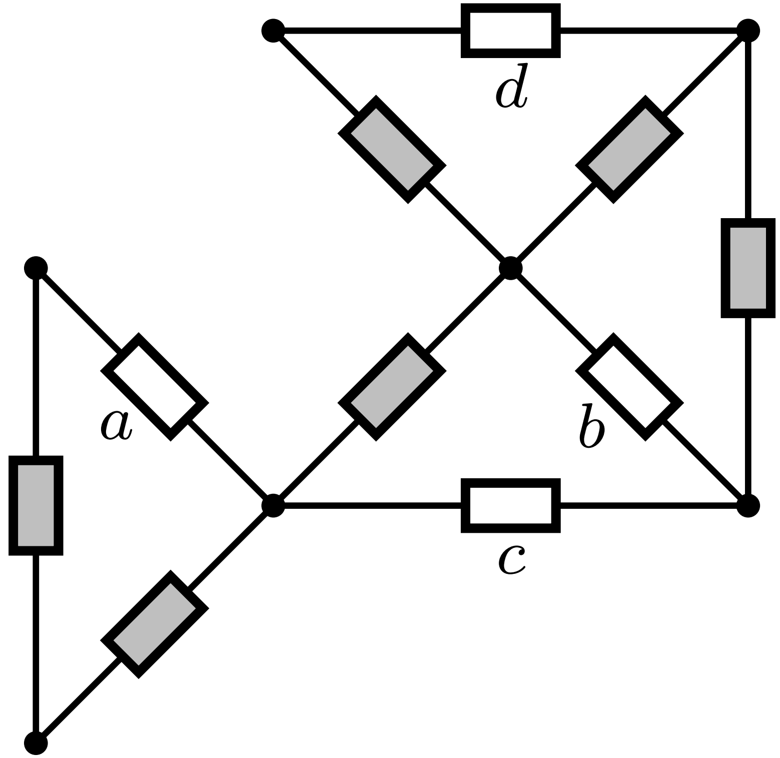


$$e = t_2 Z_k(s) f$$

$$-f = \frac{1}{(Z_a(s) + Z_b(s))} e$$

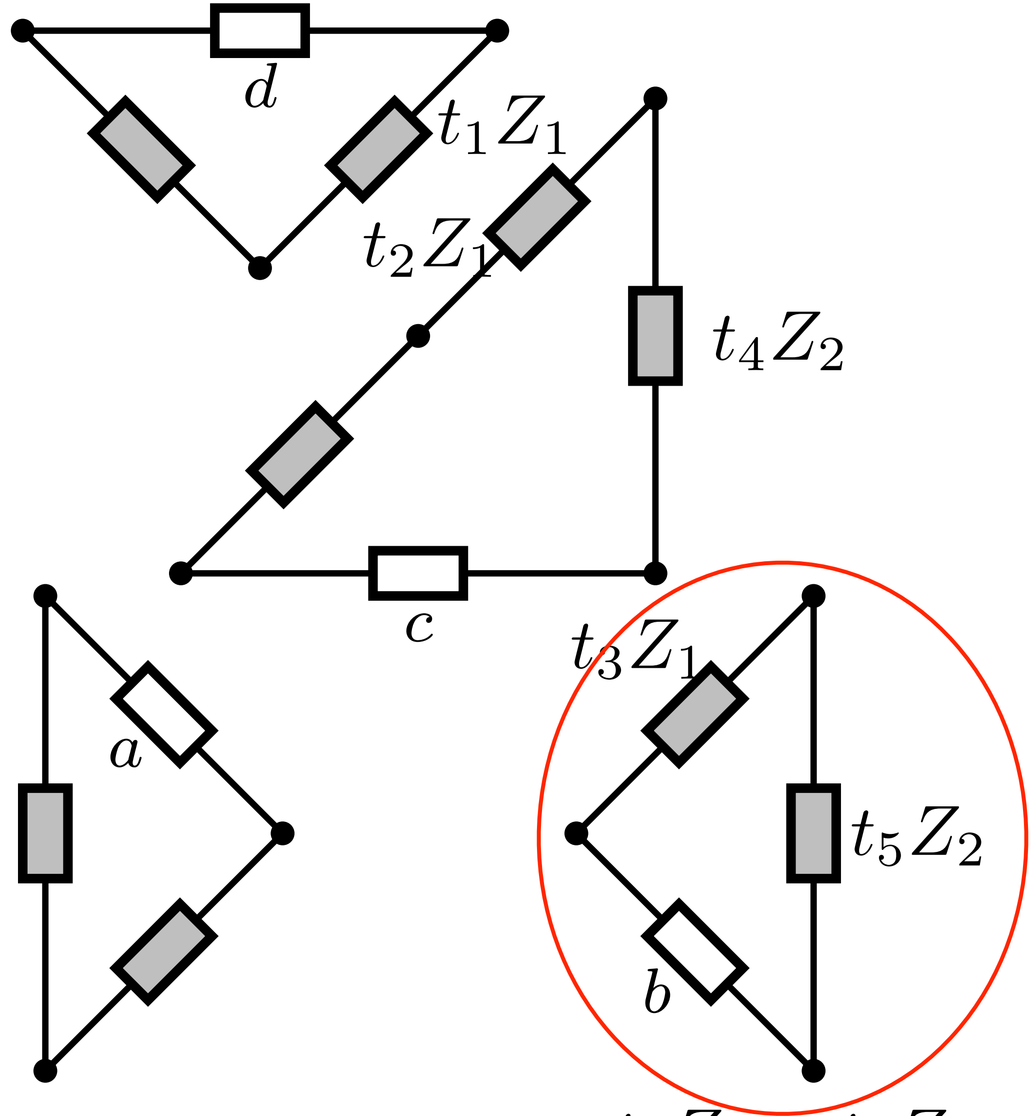
$\Longrightarrow$ use Nyquist with $L(s) = \dfrac{t_2 Z_k(s)}{Z_a(s) + Z_b(s)}$

(we will be assuming that Z_k is open circuit stable and Z_a, Z_b short circuit stable).



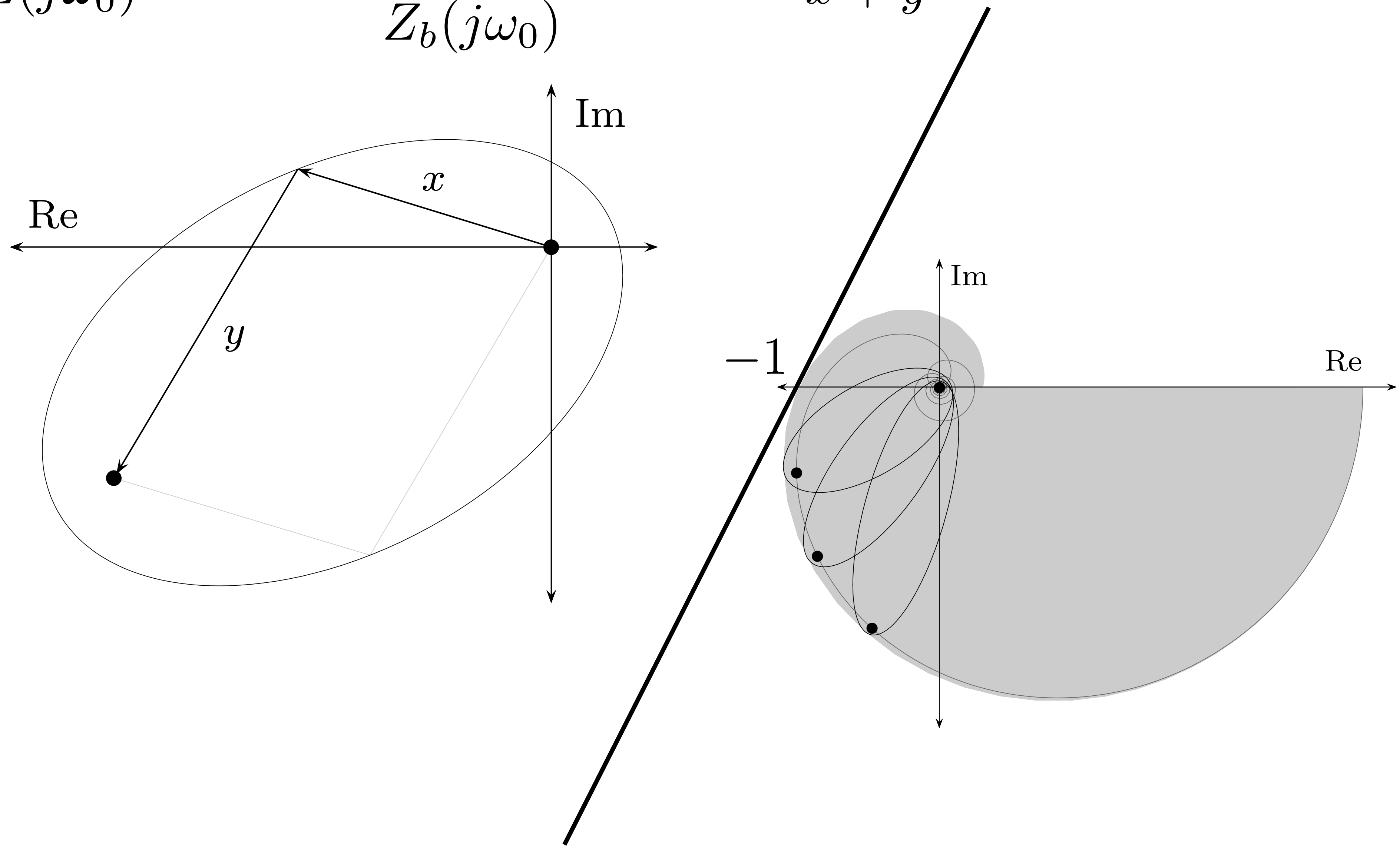
$$\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} = 1$$

$$\frac{1}{t_4} + \frac{1}{t_5} = 1$$



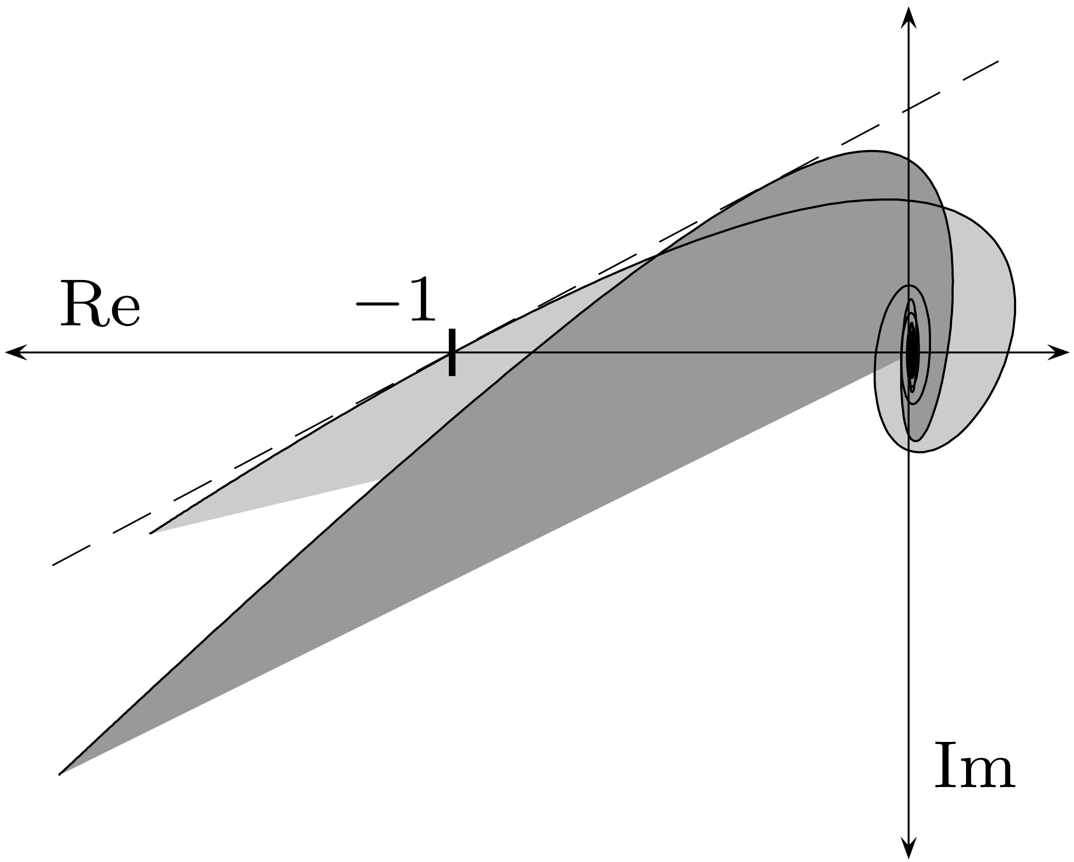
$$L(s) = \frac{t_3Z_1 + t_5Z_2}{Z_b}$$

$$L(j\omega_0) = \frac{t_3 Z_1(j\omega_0) + t_5 Z_2(j\omega_0)}{Z_b(j\omega_0)} = x + y$$



Network is stable if this *and all other* curves lies to the right of a globally specified line through -1

Can do Popov too!



So, what's going on?

Return Ratio of form GA

$$G = \begin{bmatrix} g_1 & 0 & \dots & 0 \\ 0 & g_2 & & \\ \vdots & & \ddots & \\ 0 & \dots & & g_n \end{bmatrix}, \quad [A]_{ij} \in \Re$$

If $A = A^T > 0$ then

$$\sigma(GA) \in \mathrm{Co}\{g_i\}\rho(A)$$ e.g. V (2000)

If $g_1 = g_2 = \cdots = g_n = g$ then

$$\sigma(GA) = g\sigma(A)$$ e.g. Fax & Murray (2001)

e.g $A = R\,\mathrm{diag}\{l_i\}R^T$ or $A =$ a Laplacian (consensus problems).

Can these be put together?

$$AGv = \lambda v$$
$$Gv = A^{-1}v$$
$$\implies v^*Gv = \lambda v^* A^{-1} v$$

So $W(G) \cap W(A^{-1}) = \emptyset \implies$ no eigenvalue at λ

(where $W(X) = \left\{ \frac{v^*Xv}{v^*v} : v \in \mathbb{C}^n, v \neq 0 \right\}$)

also

$$v^*G^*Gv = \lambda^2 v^* A^{-*} A^{-1} v$$

So $DW(G) \cap DW(A^{-1}) = \emptyset \implies$ no eigenvalue at λ

([Jönsson and Kao 2010, Lestas 2012])

(where $DW(X) = \left\{ \Re\frac{v^*Xv}{v^*v}, \Im\frac{v^*Xv}{v^*v}, \frac{v^*X^*Xv}{v^*v} : v \in \mathbb{C}^n, v \neq 0 \right\}$)

Also

$$DW(G) \cap DW(A^{-1}) = \emptyset \iff DW(G^{-1}) \cap DW(A) = \emptyset$$

What about neighbouring dynamics

Could consider $\sqrt{G}A\sqrt{G}$, but strongest results are in the bipartite case:

e.g. $G = \mathrm{diag}(f_1, f_2, \ldots, h_1, h_2, \ldots)$ $A = [0\ R; R^T\ 0]$, $A_{ij} \in \{-1, 0, 1\}$

$$\sigma\left(\mathrm{diag}(g_i) R^T \,\mathrm{diag}(h_i) R\right) \subset \mathrm{Co}\{m_i h_i S(n_j g_j : R_{ij} \neq 0)\}$$

where $S(X) = \mathrm{Co}(\sqrt{X})^2$ [V (2002), Lestas & V (2006)]

A better result is

$$\sigma\left(\operatorname{diag}(g_i) R^T \operatorname{diag}(h_i) R\right) \subset \mathrm{Co}\{h_i E(n_j g_j : R_{ij} \neq 0)\}$$

where $E(x_i)$ is ellipse with foci 0, $\sum x_i$ and major axis $\sum |x_i|$. [Pates & V (2012)]

proof :

$$\begin{aligned}\sigma\left(\operatorname{diag}(g_j) R^T \operatorname{diag}(h_i) R\right) &= \sigma\left(\operatorname{diag}(\sqrt{g_j}) R^T \operatorname{diag}(h_i) R \operatorname{diag}(\sqrt{g_j})\right) \\ &= \sigma\left(\sum_i \operatorname{diag}(\sqrt{g_j}) R_{i\cdot}^T h_i R_{i\cdot} \operatorname{diag}(\sqrt{g_j})\right)\end{aligned}$$

etc

Open question: Local conditions for eigenvalue locations

Richard Pates & Glenn Vinnicombe

$$G = \begin{bmatrix} g_1 & 0 & \dots & 0 \\ 0 & g_2 & & \\ \vdots & & \ddots & \\ 0 & \dots & & g_n \end{bmatrix}, \quad [A]_{ij} \in \{0, -1, 1\}$$

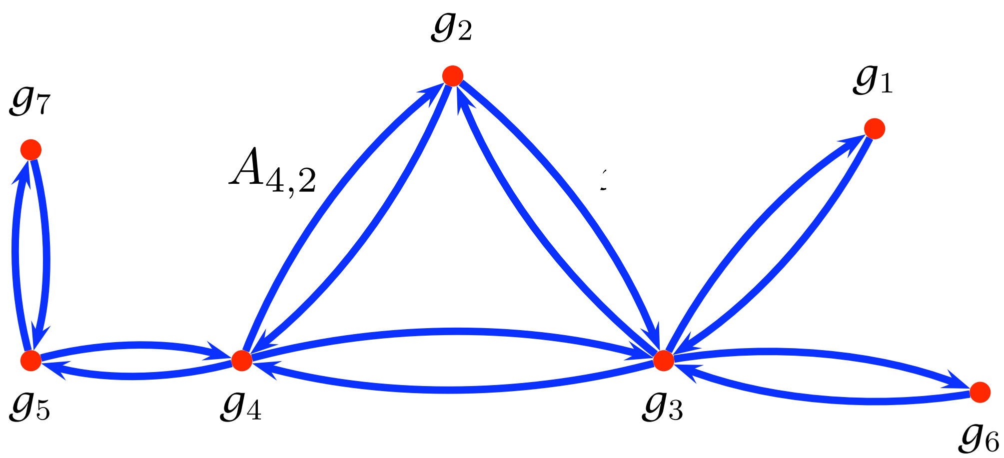
Identify A with the adjacency matrix of a *directed* graph, with $g_i \in \mathbb{C}$ labelling the nodes and R_{ij} labelling the edges.

What can we say about the $\sigma(GA)$ in terms of *local information* about the cycles of A (including those of length 2)?

Each node knows all cycles it participates in, and for each of those cycles g_i, A_{ij}, n_i along that cycle.

Is it possible, for some region $\mathcal{B}$, to come up with a yes/no question such that if all nodes say "yes", based on their local information, then $\sigma(GA) \in \mathcal{B}$?

What is the smallest $\mathcal{B}$ (and associated question).



What we know (bipartite & symmetric)

If

$$G = \mathrm{diag}(f_1, f_2, \ldots, h_1, h_2, \ldots),\ A = [0\ R; R^T\ 0], A_{ij} \in \{0, 1\}$$

$$\sigma\left(GA\right)^2 \subset \mathrm{Co}\{f_i E(n_j h_j : R_{ij} \neq 0)\}$$

where $E(x_i)$ is ellipse with foci 0, $\sum x_i$ and major axis $\sum |x_i|$, $n_j =$ in-degree of node h_j.

For $\mathcal{B}$ being the region to the right of a given line through -1,

and question is "does your ellipse lie in $\mathcal{B}$

If all answer "yes", then $\sigma(GA) \in \mathcal{B}$