Schrödinger Bridges classical and quantum evolution

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LCCC
Workshop on Dynamic and Control in Networks
Lund, October 2014

- History of Schrodinger bridges
- Bridges for Markov chains
- The Hilbert metric
- Bridges for quantum (TPTP) evolutions
- Bridges for Gauss-Markov process

Schrodinger 1931/32:
The time reversal of the laws of nature

Kolmogoroff:
The reversibility of the statistical laws of nature

Bernstein 1932
Fortet 1940
Beurling 1960
Jamison 1974/75
Follmer 1988

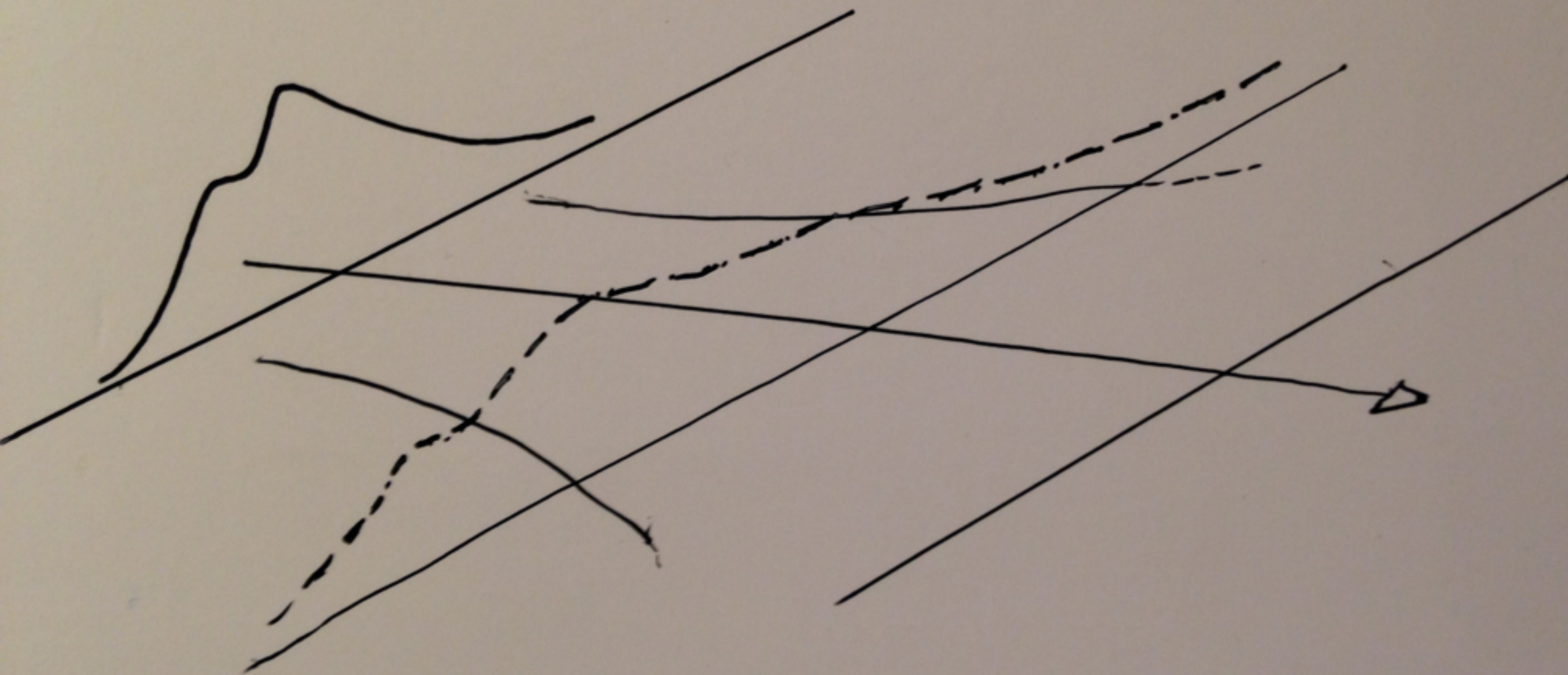
connections to Nelson’s stochastic mechanics
Zambrini, Wakolbinger, Dai Pra, Pavon,Ticozzi
and others

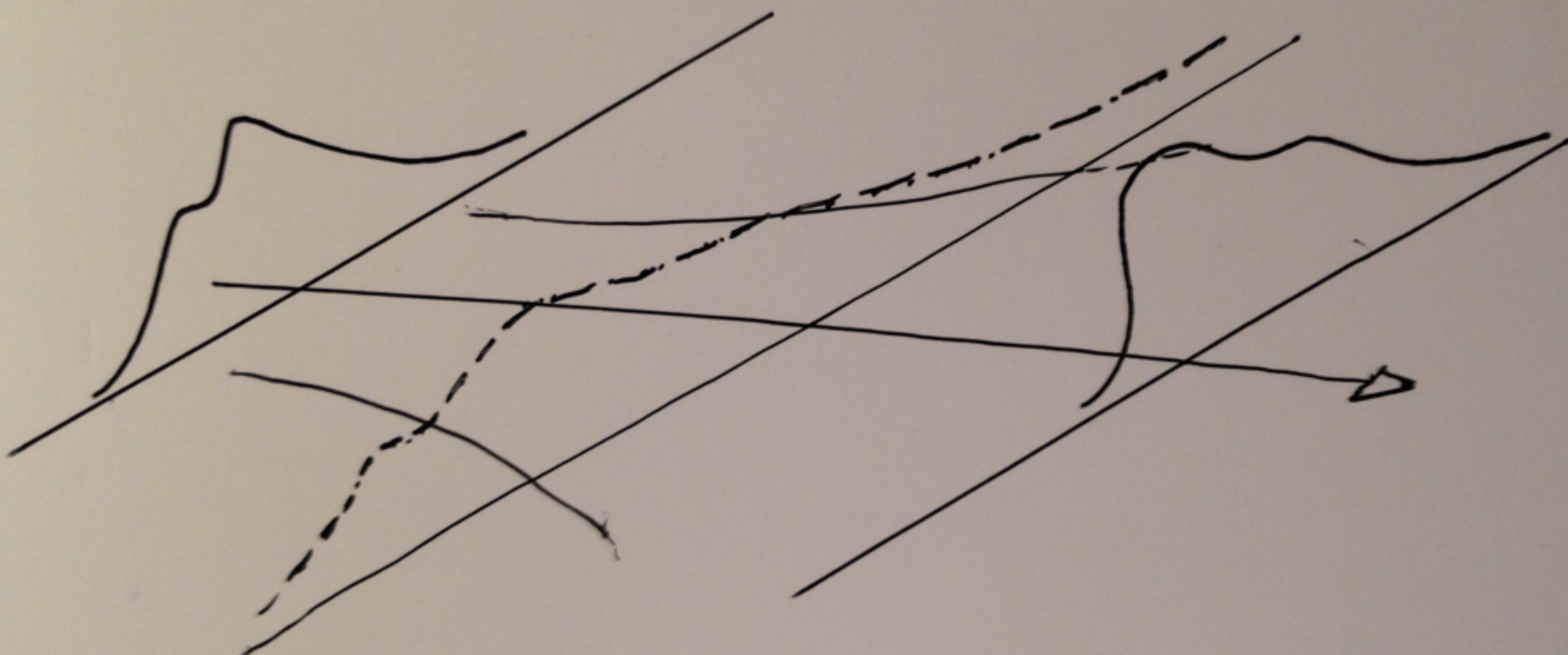
Hilbert metric:

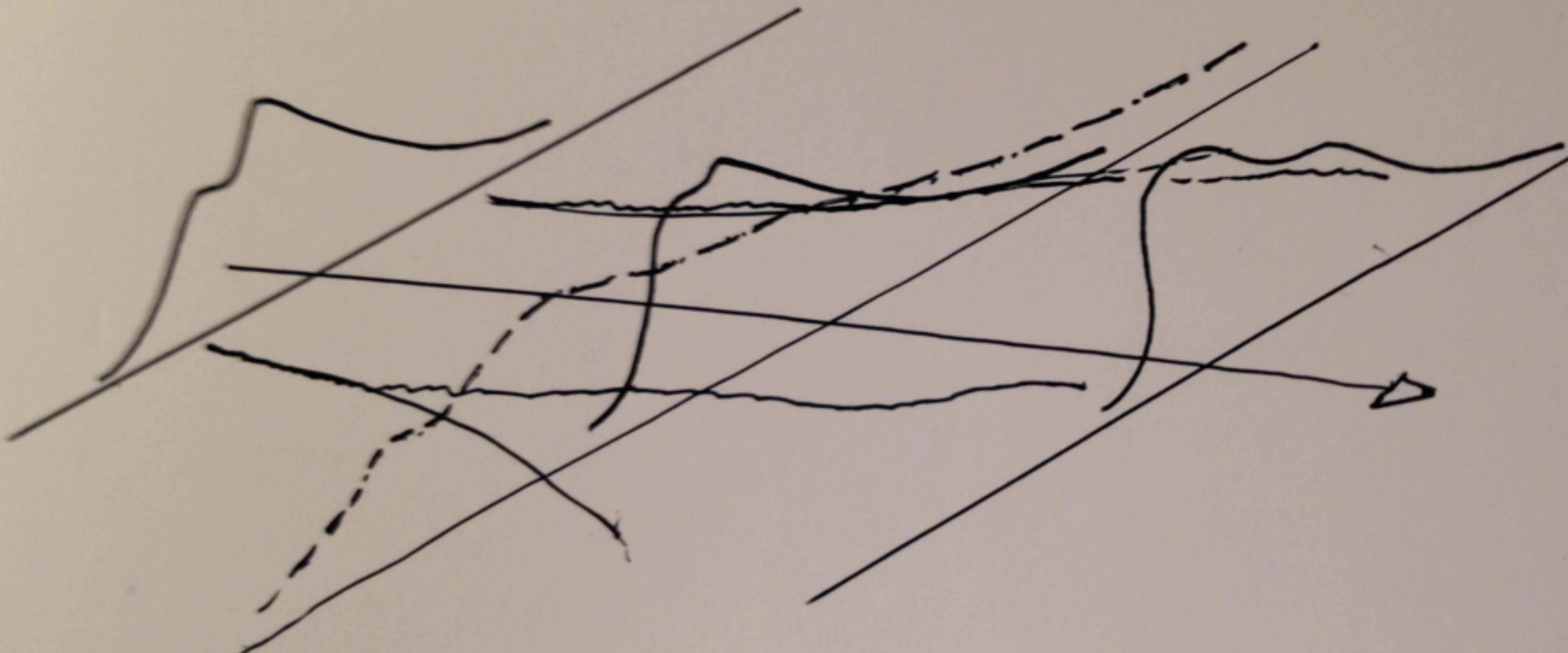
Hilbert 1895
Birkhoff 1957
Bushell 1973

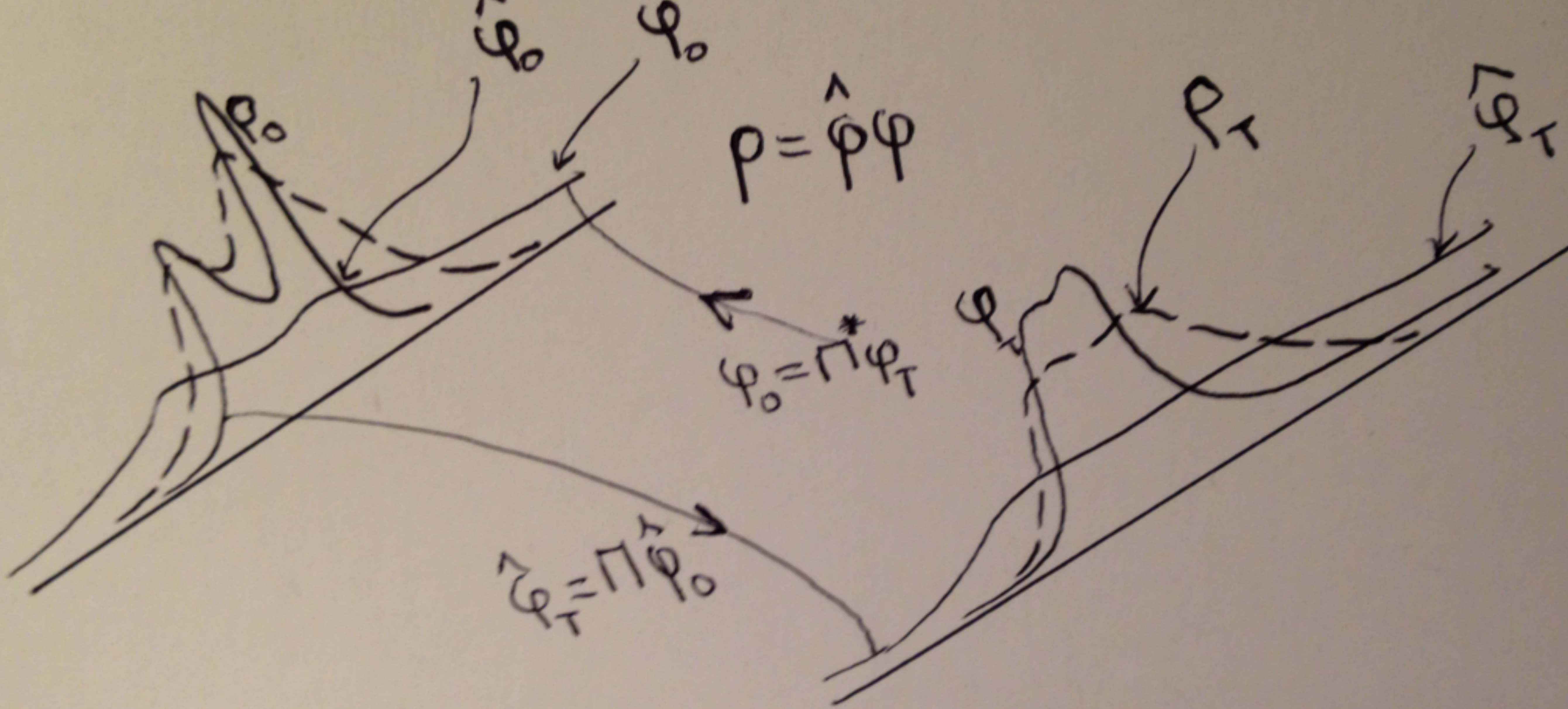
Sepulchre, Sarlette, Rouchon 2010
Reeb, Kastoryano, Wolf 2011

- Schrodinger 1931/1932: suppose a large number of Brownian particles observed at two different times to evolve between two empirical distributions. What is the most likely intermediate distribution at any point in time?








$\hat{\varphi}_0$
φ_0
ρ_0
$\rho = \hat{\varphi}\varphi$
ρ_T
$\hat{\varphi}_T$
φ_T
$\varphi_0 = \Pi^* \varphi_T$
$\hat{\varphi}_T = \Pi \hat{\varphi}_0$

Given initial and final distribution $p_0(x)$, $p_T(x)$ and transition $p(x,y)$

Schrödinger hypothesised that

$$p_T(\cdot) \neq \int p(x,\cdot)p_0(x)dx$$
$$=: \Pi_{0T}(p_0(x))$$

$\Pi_{0t} \;:\; q_0(x) \to q_t(x)$

$\frac{\partial q(t,x)}{\partial t} = \frac{1}{2}\frac{\partial^2 q(t,x)}{\partial x^2}, \quad q(0,x) = q_0(x).$

Large deviations
Sample paths
Relative entropy
Stochastic Control

Schrödinger system

discretized time, space, N-particles

Stirling's approximation

optimized, lagrange multipliers

the most likely joint density and transition probability

$$\boxed{P^\star(x_0, x_T) = \hat{\phi}(x_0)p(x_0, x_T)\phi(x_T)} \text{ and } \boxed{p^\star(x_0, x_T) = p(x_0, x_T)\frac{\phi(x_T)}{\phi(x_0)}}$$

$$p_0(x_0) = \hat{\phi}(x_0)\int p(x_0, x_T)\phi(x_T)dx_T = \hat{\phi}(x_0) \overbrace{\Pi^\dagger_{0,T} \underbrace{\phi(x_T)}_{\phi_T(x_T)}}^{\phi_0(x_0)}$$

$$p_T(x_T) = \phi(x_T)\int p(x_0, x_T)\hat{\phi}(x_0)dx_0 = \phi(x_T) \overbrace{\Pi_{0,T} \underbrace{\hat{\phi}(x_0)}_{\hat{\phi}_0(x_0)}}^{\hat{\phi}_T(x_T)}$$

and $\boxed{p_t(x_t) = \hat{\phi}_t(x_t)\phi(x_t)}$ where

$$\phi_t(x_t) := \Pi^\dagger_{t,T}\phi(x_T)$$
$$\hat{\phi}_t(x_t) := \Pi_{0,t}\hat{\phi}(x_0)$$

Schrödinger system

Schrödinger: there exists a solution "except possibly for very nasty p_0, p_T because the question leading to the pair of equations is so reasonable."

Existence/uniqueness

Fortet 1940

Résolution d'un system d'equations de M. Schr¨dinger

Beurling 1960

An automorphism of product measures

Markov chains

$\{1, \ldots, N\}$ states, $x = (x_0, x_1, \ldots, x_T)$ sample path
Π_t stochastic matrices, $t \in \{1, \ldots, T\}$
$P \in$ probability induced by Π's on $\{1, \ldots, N\}^{T+1}$

$$P(x_0, \ldots, x_T) = P(x_0, x_T) P(x_1, \ldots, x_{T-1} \mid x_0, x_T)$$

Schrödinger question

given p_0, p_T
$p_T \neq \Pi_T \cdots \Pi_1 p_0$

find
$Q(x_0, \ldots, x_T) = Q(x_0, x_T) Q(x_1, \ldots, x_{T-1} \mid x_0, x_T)$
such that
$\sum_{x_T} Q(x_0, x_T) = p_0(x_0)$
$\sum_{x_0} Q(x_0, x_T) = p_T(x_T)$
and minimizes the relative entropy

$$\sum_{all} Q \log \frac{Q}{P} = \sum_{x_0, x_T} Q(x_0, x_T) \log \frac{Q(x_0, x_T)}{P(x_0, x_T)} + \sum_{all} Q(\cdot \mid x_0, x_T) \log \frac{Q(\cdot \mid x_0, x_T)}{P(\cdot \mid x_0, x_T)} Q(x_0, x_T)$$

Lagrangian

$$L(Q) = \sum_{x_0, x_T} Q(x_0, x_T) \log \frac{Q(x_0, x_T)}{P(x_0, x_T)}$$
$$+ \sum_{x_0} \lambda(x_0) \left(\sum_{x_T} Q(x_0, x_T) - p_T(x_T) \right)$$
$$+ \sum_{x_T} \mu(x_T) \left(\sum_{x_0} Q(x_0, x_T) - p_0(x_0) \right)$$

$\lambda(x_0) \sim \hat{\phi}_0$

$\mu(x_T) \sim \phi_T$

such that with $\Pi = \Pi_T \cdots \Pi_1$

Schrödinger system

$$\hat{\phi}_T = \Pi \hat{\phi}_0$$
$$\phi_0 = \Pi^\dagger \phi_T$$
$$p_0 = \phi_0 \circ \hat{\phi}_0$$
$$p_T = \phi_T \circ \hat{\phi}_T$$

if there is a solution

$$\Pi^\star = D_{\phi_T} \Pi D_{\phi_0}^{-1}$$
$$[Q(x_0, x_T)]_{x_0, x_T} = D_{\phi_T} \Pi D_{\hat{\phi}_0}$$

Hilbert metric

S real Banach space

K closed solid cone in S

$$x \preceq y \Leftrightarrow y - x \in K,$$

$$\begin{aligned} M(x,y) &:= \inf\{\lambda \mid x \preceq \lambda y\} \\ m(x,y) &:= \sup\{\lambda \mid \lambda y \preceq x\}. \end{aligned}$$

define the Hilbert metric:

$$d_H(x,y) := \log\left(\frac{M(x,y)}{m(x,y)}\right).$$

Examples:

i) positive cone in $\mathbb{R}^n$

ii) positive definite Hermitian matrices

d_H-gain bound of positive maps

Π is a positive map:

$$\Pi \ : \ K\backslash\{0\} \rightarrow K\backslash\{0\}.$$

Projective diameter

$$\Delta(\Pi) := \sup\{d_H(\Pi(x), \Pi(y)) \mid x, y \in K\backslash\{0\}\}$$

Contraction ratio, or gain/H-norm

$$\|\Pi\|_H := \inf\{\lambda \mid d_H(\Pi(x), \Pi(y)) \leq \lambda d_H(x, y), \text{ for all } x, y \in K\backslash\{0\}\}.$$

Birkhoff-Bushell theorem

Let Π positive, monotone, homogeneous of degree m, i.e.,

$$x \preceq y \Rightarrow \Pi(x) \preceq \Pi(y),$$

and

$$\Pi(\lambda x) = \lambda^m \Pi(x),$$

then

$$\|\Pi\|_H \leq m.$$

For the special case where Π is also linear, the (possibly stronger) bound

$$\|\Pi\|_H = \tanh(\frac{1}{4}\Delta(\Pi))$$

also holds.

Solution of the Schrödinger system

Lemma

Let $\Pi >_e 0$ (element-wise positive) stochastic matrix

p_0, p_T probability vectors

then $\|\Pi\|_H < 1$.

proof

i) $\Delta(\Pi) = \sup\{d_H(\Pi(x), \Pi(y)) \mid x, y \in K \backslash \{0\}\}$

remains the same if we restrict x, y

to be probability vectors

ii) $d_H(\Pi(x), \Pi(y)) < \infty \; \forall x, y$.

iii) the probability simplex is compact.

Solution of the Schrödinger system

Consider

$$\begin{array}{ccccc} & \hat{\varphi}_0 & \xrightarrow{\Pi} & \hat{\varphi}_T & \\ \hat{\varphi}_0(x_0) = \frac{\mathbf{p}_0(x_0)}{\varphi_0(x_0)} & \uparrow & & \downarrow & \varphi_T(x_T) = \frac{\mathbf{p}_T(x_T)}{\hat{\varphi}_T(x_T)} \\ & \varphi_0 & \xleftarrow{\Pi^\dagger} & \varphi_T & \end{array}$$

where

$$\mathcal{D}_T \; : \; \varphi_0 \; \mapsto \; \hat{\varphi}_0(x_0) = \frac{\mathbf{p}_0(x_0)}{\varphi_0(x_0)}$$

$$\mathcal{D}_T \; : \; \hat{\varphi}_T \; \mapsto \; \varphi_T(x_T) = \frac{\mathbf{p}_T(x_N)}{\hat{\varphi}_T(x_T)}$$

are componentwise division of vectors $\Rightarrow d_H$-isometries!

The composition

$$\hat{\varphi}_0 \xrightarrow{\Pi} \hat{\varphi}_T \xrightarrow{\mathcal{D}_T} \varphi_T \xrightarrow{\Pi^\dagger} \varphi_0 \xrightarrow{\mathcal{D}_0} (\hat{\varphi}_0)_{\text{next}}$$

is strictly contractive in the Hilbert metric.

Sinkhorn's theorem

If $\Pi >_e 0$,
then $\exists\ a_i,\ b_j$
such that $[\pi_{ij} a_i b_j]_{i,j}$ doubly stochastic.

Cf. $p_0 = \mathbb{1}$, $p_T = \mathbb{1}$
$\Pi^\star = D_{\phi_T} \Pi D_{\phi_0}^{-1}$ doubly stochastic

i.e., $(\Pi^\star)^\dagger \mathbb{1} = \mathbb{1}$
but also $(\Pi^\star)\mathbb{1} = \mathbb{1}$

Quantum analogues

Density matrices: $\mathfrak{D} = \{\rho \geq 0 \mid \mathrm{trace}(\rho) = 1\}$

TPTP: $\mathcal{E} \;:\; \mathfrak{D} \to \mathfrak{D} \;:\; \rho \longrightarrow \sigma = \sum_{i=1}^{n_{\mathcal{E}}} E_i \rho E_i^{\dagger}$
with

$$\sum_{i=1}^{n_{\mathcal{E}}} E_i^{\dagger} E_i = I$$

i.e., $\mathcal{E}^{\dagger}(I) = I$

$\mathcal{E}$ is positivity improving: if $\rho \geq 0 \Rightarrow \mathcal{E}(\rho) > 0$

Reference quantum evolution

TPCP maps $\{\mathcal{E}_t; 0 \leq t \leq T-1\}$
with Kraus representation

$$\mathcal{E}_t : \sigma_t \mapsto \sigma_{t+1} = \sum_i E_{t,i}\sigma_t E_{t,i}^\dagger, \quad t = 0, 1, \ldots, T-1.$$

Consider the composition

$$\mathcal{E}_{0:T} := \mathcal{E}_{T-1} \circ \cdots \circ \mathcal{E}_0.$$

initial and a final ρ_0 and ρ_T

Problem

Find $\mathcal{F}_{0:T} = \mathcal{F}_{T-1} \circ \cdots \circ \mathcal{F}_0$ such that

$$\mathcal{F}_{0:T}(\rho_0) = \rho_T.$$

and $\mathcal{F}$ "close to" $\mathcal{E}$

“rank-1” corrections

$$\mathcal{F}_t(\cdot) = \chi_{t+1}\left(\mathcal{E}_t(\chi_t^{-1}(\cdot)\chi_t^{-\dagger})\right)\chi_{t+1}^{\dagger}$$

i.e., $\mathcal{F}_t = \Phi_{t+1} \circ \mathcal{E}_t \circ \Phi_t^{-1}$ where

Φ are rank-1 Kraus maps, $n_\Phi = 1$

Corresponds to the commutative case via: $\chi^{\dagger}\chi = \phi$

Quantum version of Sinkhorn's thm

Suppose $\mathcal{E}_{0:T}$ is positivity improving
Then, $\exists$ observables ϕ_0, ϕ_T such that,
for any factorization

$$\begin{aligned} \phi_0 &= \chi_0^\dagger \chi_0, \text{ and} \\ \phi_T &= \chi_T^\dagger \chi_T, \end{aligned}$$

the map

$$\mathcal{F}(\cdot) := \chi_T \left(\mathcal{E}_{0:T}(\chi_0^{-1}(\cdot)\chi_0^{-\dagger}) \right) \chi_T^\dagger$$

is a *doubly stochastic* Kraus map,
in that $\mathcal{F}(I) = I$ as well as $\mathcal{F}^\dagger(I) = I$.

Proof

$$\begin{array}{ccccc} & \hat{\phi}_0 & \xrightarrow{\mathcal{E}_{0,T}} & \hat{\phi}_T & \\ \hat{\phi}_0 = \phi_0^{-1} & \uparrow & & \downarrow & \phi_T = \hat{\phi}_T^{-1} \\ & \phi_0 & \xleftarrow{\mathcal{E}_{0,T}^{\dagger}} & \phi_T & \end{array}$$

The composition map

$$\mathcal{C} \;:\; \left(\hat{\phi}_0\right)_{\text{starting}} \xrightarrow{\mathcal{E}_{0,T}} \hat{\phi}_T \xrightarrow{(\cdot)^{-1}} \phi_T \xrightarrow{\mathcal{E}_{0,T}^{\dagger}} \phi_0 \xrightarrow{(\cdot)^{-1}} \left(\hat{\phi}_0\right)_{\text{next}}$$

is strictly contractive

the steps are identical

General case

Given $\mathcal{E}_{0:T}^{\dagger}$ and ρ_0 and ρ_T
if $\exists\ \phi_0,\ \phi_T,\ \hat{\phi}_0,\ \hat{\phi}_T$ solving

$$\begin{aligned}
\mathcal{E}_{0:T}^{\dagger}(\phi_T) &= \phi_0, \\
\mathcal{E}_{0:T}(\hat{\phi}_0) &= \hat{\phi}_T, \\
\rho_0 &= \chi_0 \hat{\phi}_0 \chi_0^{\dagger}, \\
\rho_T &= \chi_T \hat{\phi}_T \chi_T^{\dagger}.
\end{aligned}$$

Then, for any factorization

$$\begin{aligned}
\phi_0 &= \chi_0^{\dagger}\chi_0, \ \text{and} \\
\phi_T &= \chi_T^{\dagger}\chi_T,
\end{aligned}$$

the map

$$\mathcal{F}(\cdot) := \chi_T \left(\mathcal{E}_{0:T}(\chi_0^{-1}(\cdot)\chi_0^{-\dagger}) \right) \chi_T^{\dagger}$$

is a quantum bridge for $(\mathcal{E}_{0:T}^{\dagger}, \rho_0, \rho_T)$, namely $\mathcal{F}(I) = I$ and $\mathcal{F}^{\dagger}(\rho_0) = \rho_T$.

Conjecture

The quantum Schrödinger system has a solution
for arbitrary ρ_0, ρ_T

Snag in the proof:
$\phi \to \hat{\phi}$ and $\hat{\phi} \to \phi$ are not isometries, e.g.,

$$D_T \ : \ \hat{\phi}_T \mapsto \phi_T = \left(\rho_T^{1/2}\left(\rho_T^{-1/2}\hat{\phi}^{-1}\rho_T^{-1/2}\right)^{1/2}\rho_T^{1/2}\right)^2$$

$$\hat{D}_0 \ : \ \phi_0 \mapsto \hat{\phi}_0 = (\phi_0)^{1/2}\rho(\phi_0)^{1/2}$$

Extensive numerical evidence that
the composition has a fixed point
Software for numerical experimentation
`http://www.ece.umn.edu/~georgiou/papers/schrodinger_bridge/`

Pinned bridge

$\mathcal{E}_{0:T}$ positivity improving and two pure states

$$\rho_0 = v_0 v_0^\dagger \text{ and } \rho_T = v_T v_T^\dagger$$

(i.e., v_0, v_T are unit norm vectors), define

$$\begin{aligned} \phi_0 &:= \mathcal{E}(v_T v_T^\dagger) \\ \phi_T &:= v_T v_T^\dagger, \end{aligned}$$

and

$$\mathcal{F}^\dagger(\cdot) := \phi_T^{1/2} \mathcal{E}^\dagger(\phi_0^{-1/2}(\cdot)\phi_0^{-1/2})\phi_T^{1/2}$$

(where, clearly, $\phi_T^{1/2} = \phi_T = v_T v_T^\dagger$). Then, $\mathcal{F}^\dagger$ is TPTP and satisfies the marginal conditions

$$\rho_T = \mathcal{F}^\dagger(\rho_0).$$

Example

$$\mathcal{E}(\cdot) = E_1(\cdot)E_1^\dagger + E_2(\cdot)E_2^\dagger + E_3(\cdot)E_3^\dagger$$

$$E_1 = \begin{bmatrix} \sqrt{\frac{1}{2}} & 0 \\ 0 & 0 \end{bmatrix}, \; E_2 = \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{\frac{1}{2}} \end{bmatrix}, \; E_3 = \begin{bmatrix} 0 & \sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{2}} & 0 \end{bmatrix}.$$

$$\rho_0 = \begin{bmatrix} 1/4 & 0 \\ 0 & 3/4 \end{bmatrix} \text{ and } \rho_1 = \begin{bmatrix} 2/3 & 0 \\ 0 & 1/3 \end{bmatrix}$$

$$\phi_0 = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$$

$$\phi_1 = \begin{bmatrix} 2/3 & 0 \\ 0 & 1/3 \end{bmatrix}$$

$$\hat{\phi}_0 = \begin{bmatrix} 1/2 & 0 \\ 0 & 3/2 \end{bmatrix}$$

$$\hat{\phi}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$F_1 = \begin{bmatrix} \sqrt{2/3} & 0 \\ 0 & 0 \end{bmatrix}, \;\; F_2 = \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{1/3} \end{bmatrix}, \;\; F_3 \begin{bmatrix} 0 & \sqrt{2/3} \\ \sqrt{1/3} & 0 \end{bmatrix}$$

Recap

Hilbert metric $\Rightarrow$ constructive existence proofs for

i) classical Schrödinger systems

ii) quantum Sinkhorn version (uniform marginals)

iii) general case open

Final topic:

Schrödinger bridges for "degenerate" classical linear stochastic systems

$\equiv$ a new type of optimal control problem

Optimal steering of state-densities

min relative entropy $\leftrightarrow^{\text{Girsanov}}\leftrightarrow$ minimum energy stochastic control

$dx = bdt + dw$ diffusion

$dx = (b + u)dt + dw$ controlled diffusion

$\min\{E\{\|u\|^2\} \mid p_0, p_T\} \sim$ relative entropy from prior (dai Pra)

our interest:

inertial particles, cooling of oscillators

$dx = vdt$

$dv = (b + u)dt + dw$ controlled degenerate diffusion

Optimal steering of state-densities

$$dx(t) = A(t)x(t)dt + B(t)u(t)dt + B(t)dw(t)$$

Given initial and terminal (target) Gaussian densities with covariances Σ_0, Σ_T.

Find $u(t)$ with $t \in [0, T]$ that steers the system from the initial to the target state density and minimizes

$$E\{\int_0^T u(t)'u(t)dt\}$$

Optimal steering of state-densities

Theorem (Gauss-Markov Schrödinger bridge):
There exists a unique solution to the following (analogue of the Schrödinger system)

$Q(T)$, $P(0)$ values for matrices satisfying

$\Sigma_0^{-1} = Q(0)^{-1} + P(0)^{-1}$

$\Sigma_T^{-1} = Q(T)^{-1} + P(T)^{-1}$

and $Q(0)$, $P(T)$ obtained via

$\dot{Q}(t) = A(t)Q(t) + Q(t)A(t)' + B(t)B(t)'$

$\dot{P}(t) = A(t)P(t) + P(t)A(t)' - B(t)B(t)'$

with $Q(t)$ invertible over $[0, T]$.

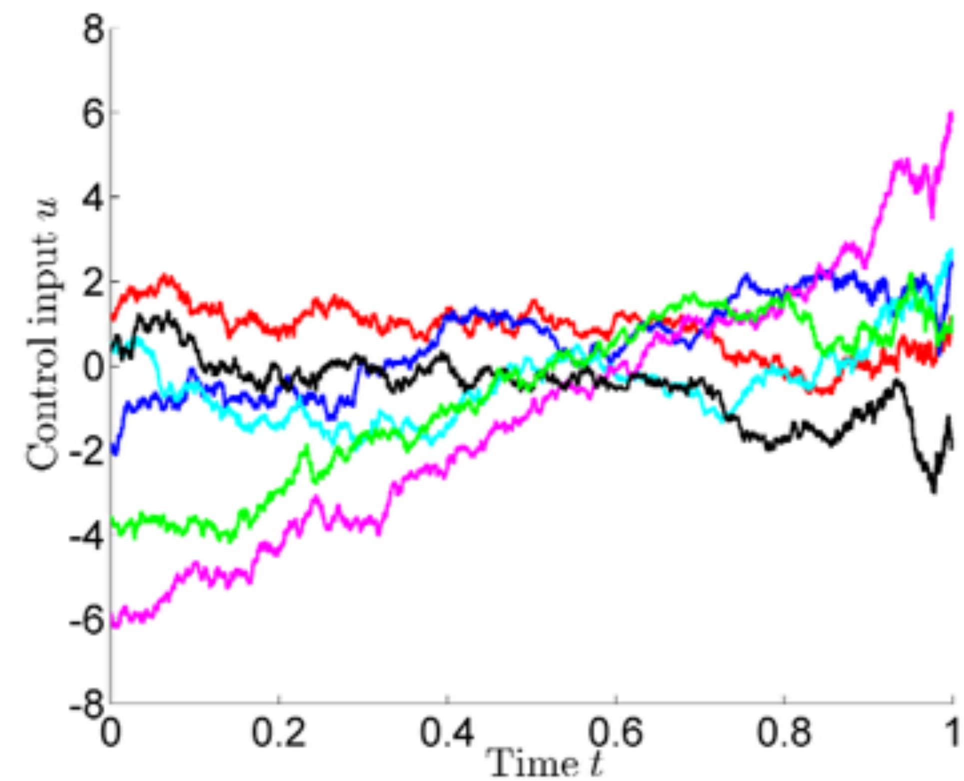
The optimal control is $u(t) = -B(t)'Q(t)^{-1}x(t)$
The controlled degenerate diffusion is the closest to the uncontrolled diffusion in the relative entropy sense.

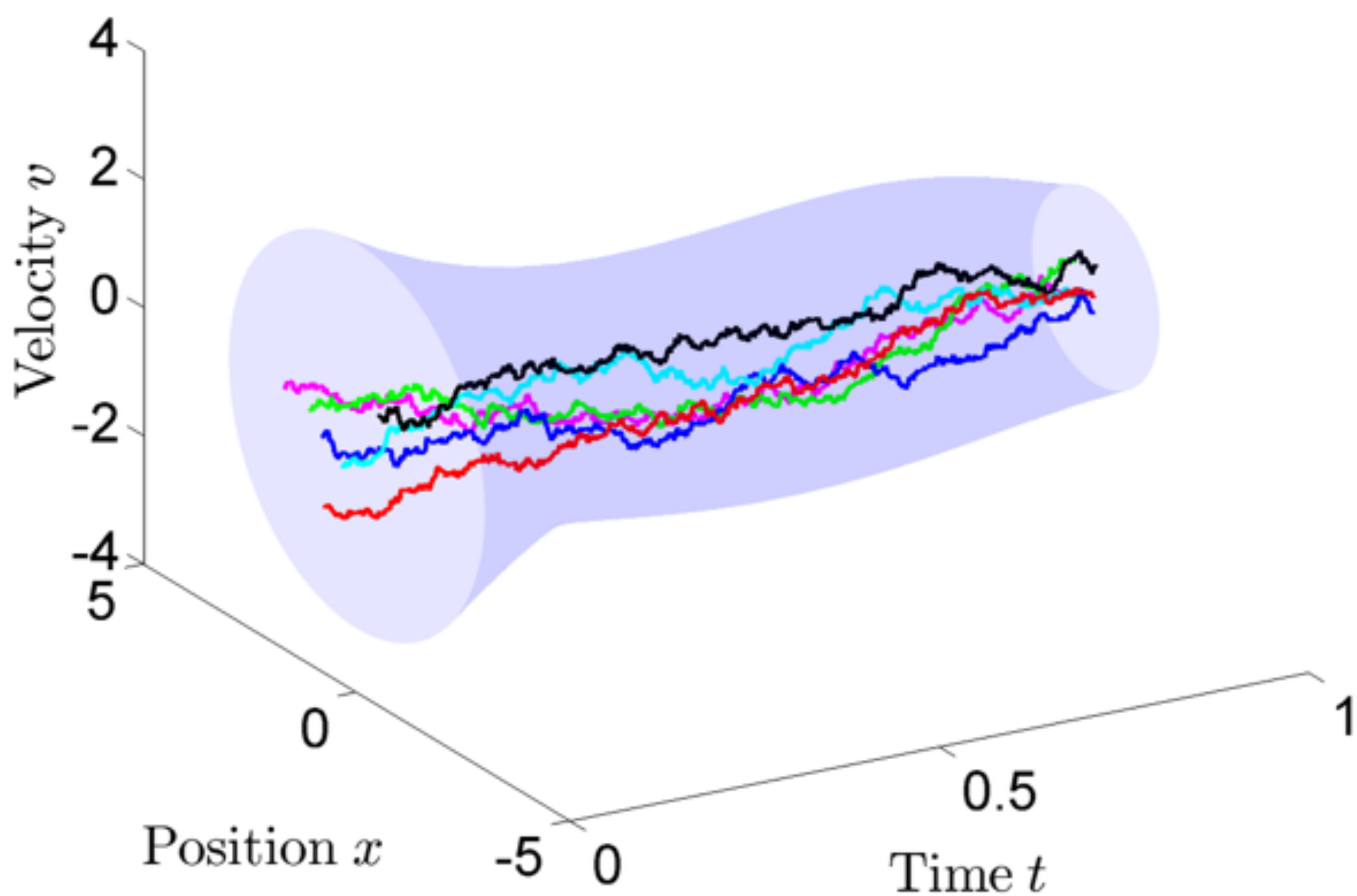
$$Q(0) = N(T,0)^{1/2} S_0^{1/2} \left(S_0 + \frac{1}{2} I - \left(S_0^{1/2} S_T S_0^{1/2} + \frac{1}{4} I \right)^{1/2} \right)^{-1} S_0^{1/2} N(T,0)^{1/2}$$

$N(T,0)$ is the controllability Grammian.

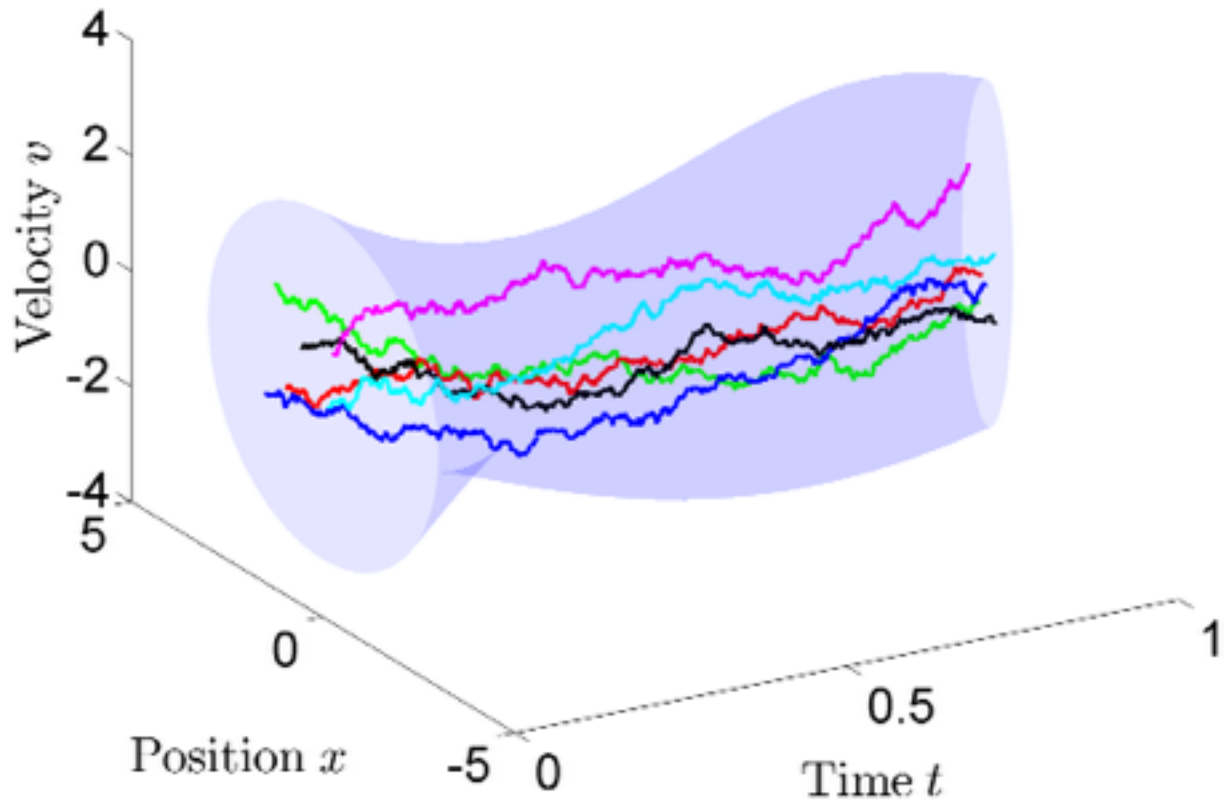
Gauss Markov model for inertial particles

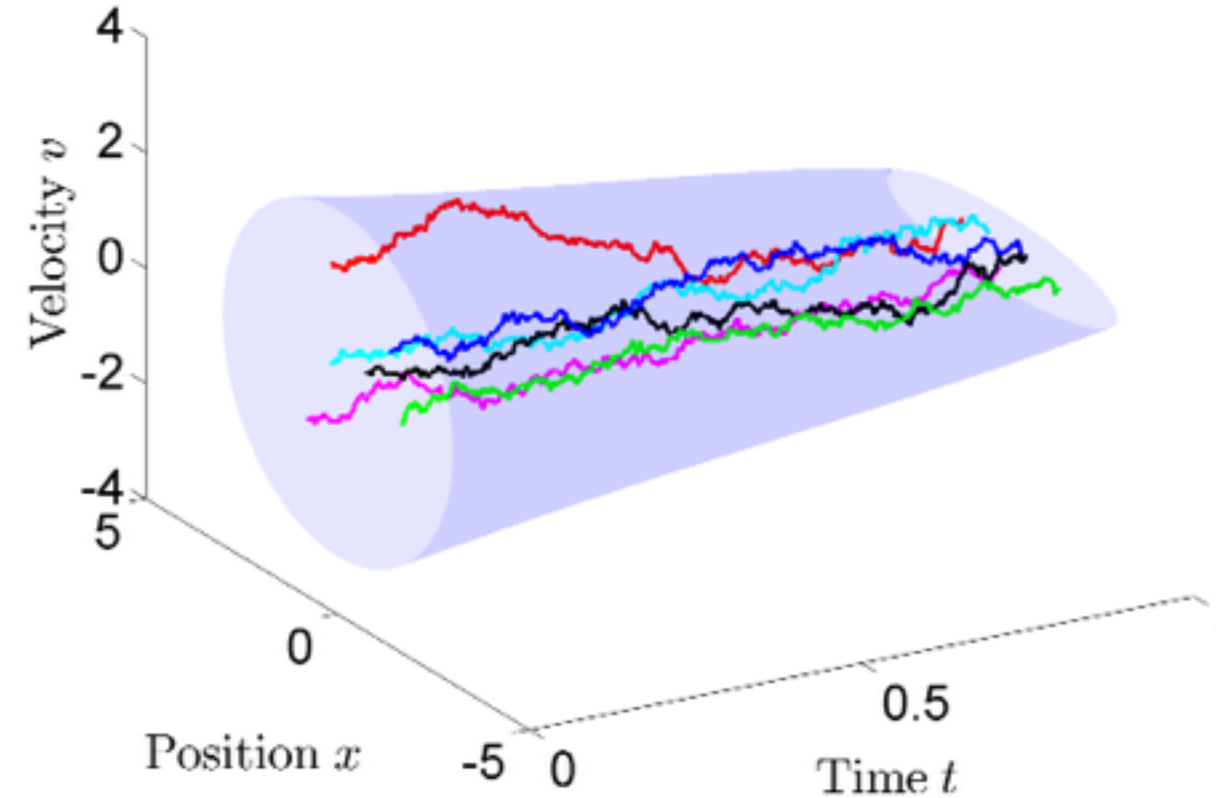
$$\begin{aligned} dx(t) &= v(t)dt \\ dv(t) &= u(t)dt + dw(t) \end{aligned}$$





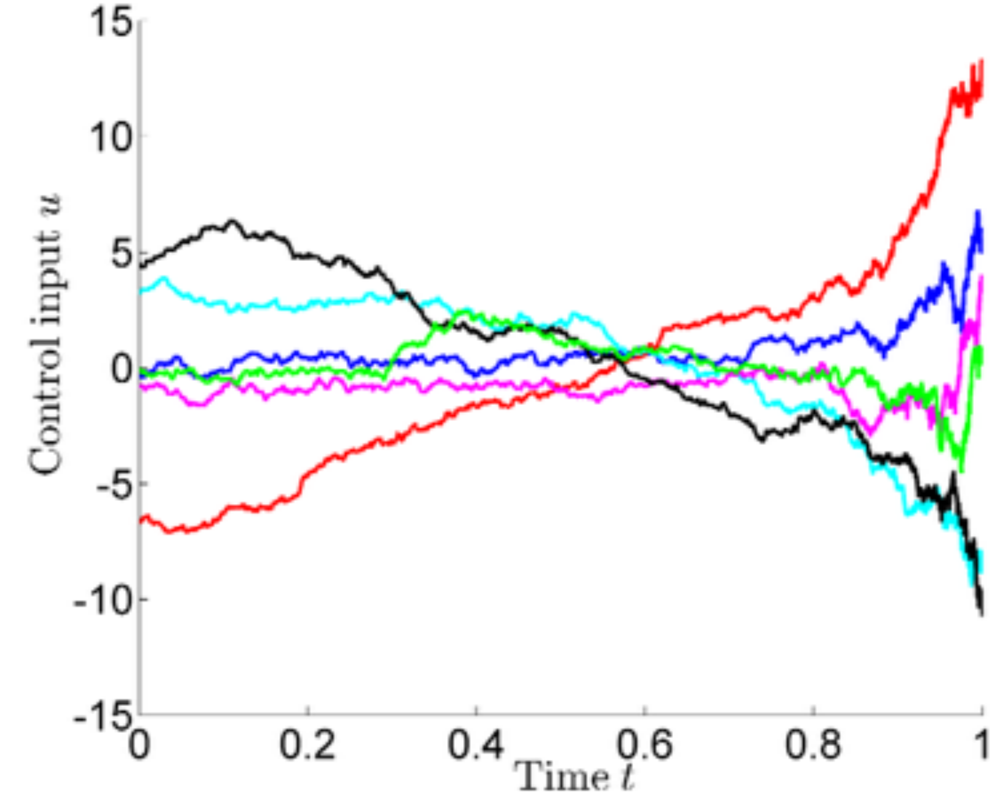
Gauss Markov model for inertial particles

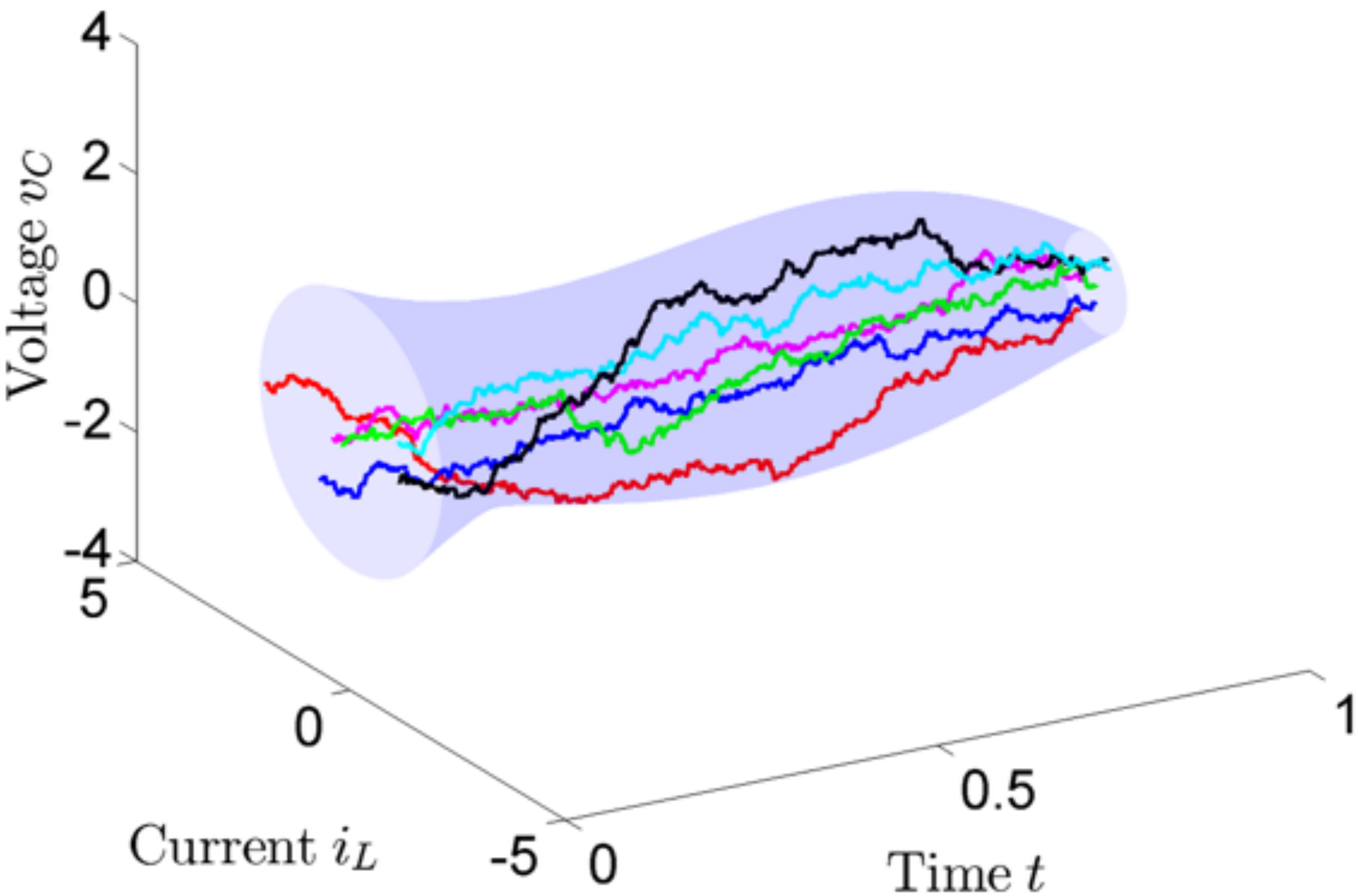


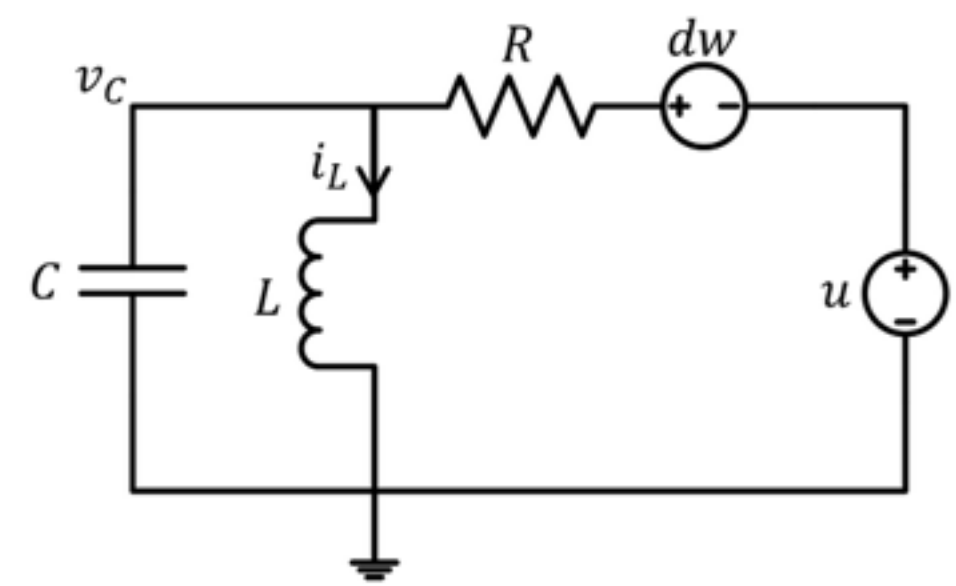


Gauss Markov model
for Nyquist-Johnson noise driven oscillator

$$
\begin{aligned}
L di_L(t) &= v_C(t)dt \\
RC dv_C(t) &= -v_C(t)dt - Ri_L(t)dt + u(t)dt + dw(t)
\end{aligned}
$$







Gauss Markov model for inertial particles: state-cost ~ particles with losses

$$dX(t) = f(X(t),t)dt + \sigma(X(t),t)dw(t)$$

$$\inf_{(\tilde\rho,\tilde u)} \int_{\mathbb{R}^N}\int_0^T \left[\frac{1}{2}\|u\|^2 + V(x,t)\right]\tilde\rho(x,t)dtdx,$$

$$\frac{\partial\tilde\rho}{\partial t} + \nabla\cdot((f+\sigma u)\tilde\rho) = \frac{1}{2}\sum_{i,j=1}^N \frac{\partial^2(a_{ij}\tilde\rho)}{\partial x_i\partial x_j},$$

$$a_{ij}(x,t) = \sum_k \sigma_{ik}(x,t)\sigma_{kj}(x,t)$$

$$\tilde\rho(0,x) = \rho_0(x), \quad \tilde\rho(T,y) = \rho_T(y).$$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(f(x,t)\rho) + V(x,t)\rho = \frac{1}{2}\sum_{i,j=1}^N \frac{\partial^2(a_{ij}(x,t)\rho)}{\partial x_i\partial x_j}$$

Schrödinger system

$$
\begin{aligned}
\frac{\partial \varphi}{\partial t} + f(x,t)\cdot\nabla\varphi + \frac{1}{2}\sum_{i,j=1}^{N} a_{ij}\frac{\partial^2\varphi}{\partial x_i \partial x_j} &= V\varphi, \\
\frac{\partial \hat{\varphi}}{\partial t} + \nabla\cdot(f(x,t)\hat{\varphi}) - \frac{1}{2}\sum_{i,j=1}^{N} \frac{\partial^2 (a_{ij}\hat{\varphi})}{\partial x_i \partial x_j} &= -V\hat{\varphi},
\end{aligned}
$$

$$
\varphi(x,0)\hat{\varphi}(x,0) = \tilde{\rho}_0(x), \quad \varphi(x,T)\hat{\varphi}(x,T) = \tilde{\rho}_T(x)
$$

$$
\begin{aligned}
u^*(x,t) &= \sigma'\nabla\log\varphi(x,t), \\
\frac{\partial \tilde{\rho}}{\partial t} + \nabla\cdot((f + a\nabla\log\varphi)\tilde{\rho}) &= \frac{1}{2}\sum_{i,j=1}^{N} \frac{\partial^2 (a_{ij}\tilde{\rho})}{\partial x_i \partial x_j},
\end{aligned}
$$

Controllability of Fokker-Planck - Linear-Gaussian

$$dx(t) = Ax(t)dt + Bu(t)dt + B_1 dw(t)$$
$$\text{with } x(0) = x_0 \text{ a.s.}$$

Thm: (A,B) controllable is sufficient to steer the system from any initial Gaussian distribution to a final one at t=T.

Thm: A Gaussian state-pdf can be "sustained" with constant state-feedback iff the state covariance satisfies

$$(A - BK)\Sigma + \Sigma(A' - K'B') + B_1 B_1' = 0.$$

equivalently, $\text{rank}\begin{bmatrix} A\Sigma + \Sigma A' + B_1 B_1' & B \\ B & 0 \end{bmatrix} = \text{rank}\begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix}$

Compare with conditions for:
i) steering the system to a given state - controllability
ii) steering within the positive cone?
iii) maintaining the state at a given value

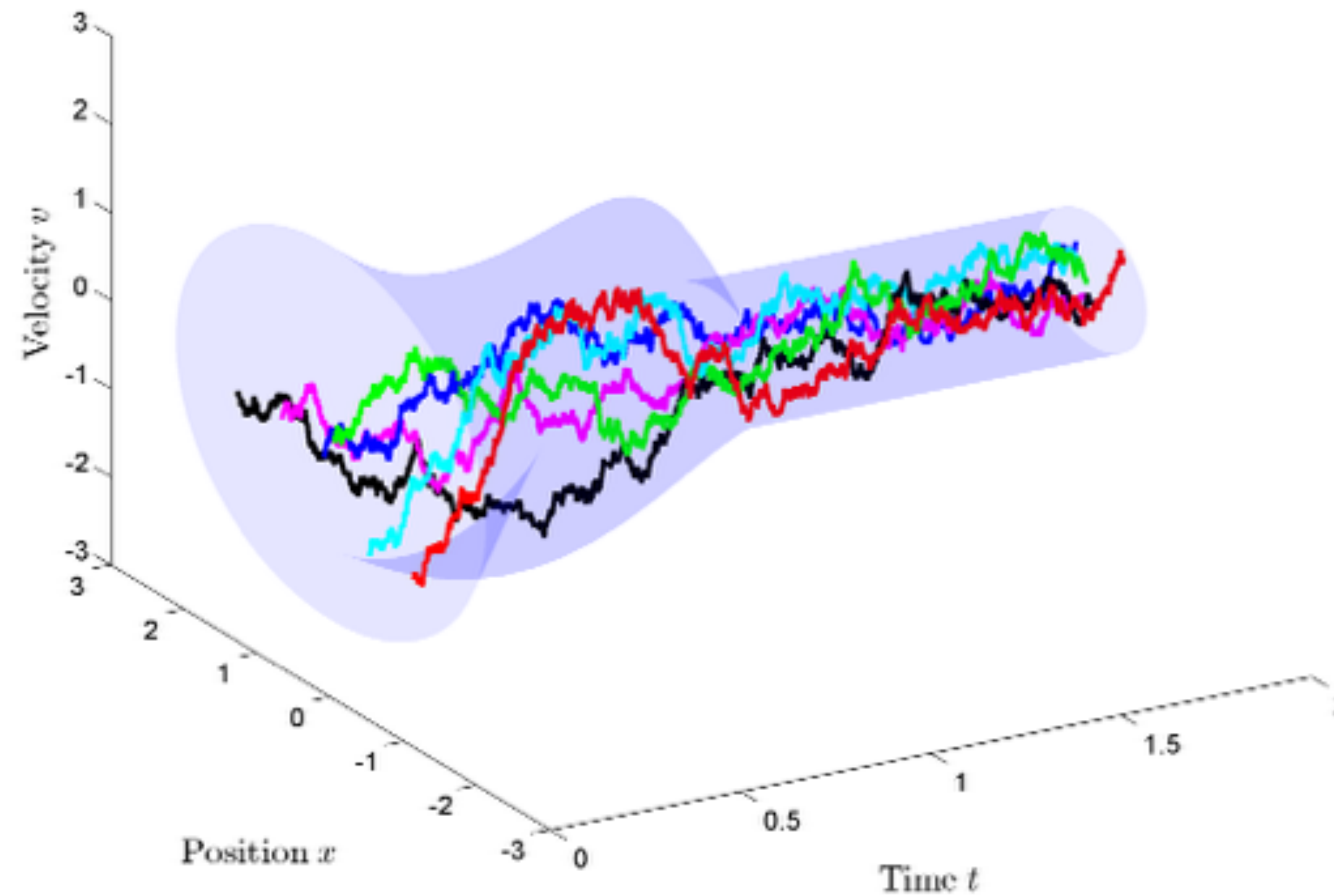
$$\dot{x}(t) = Ax(t) + Bu(t)$$

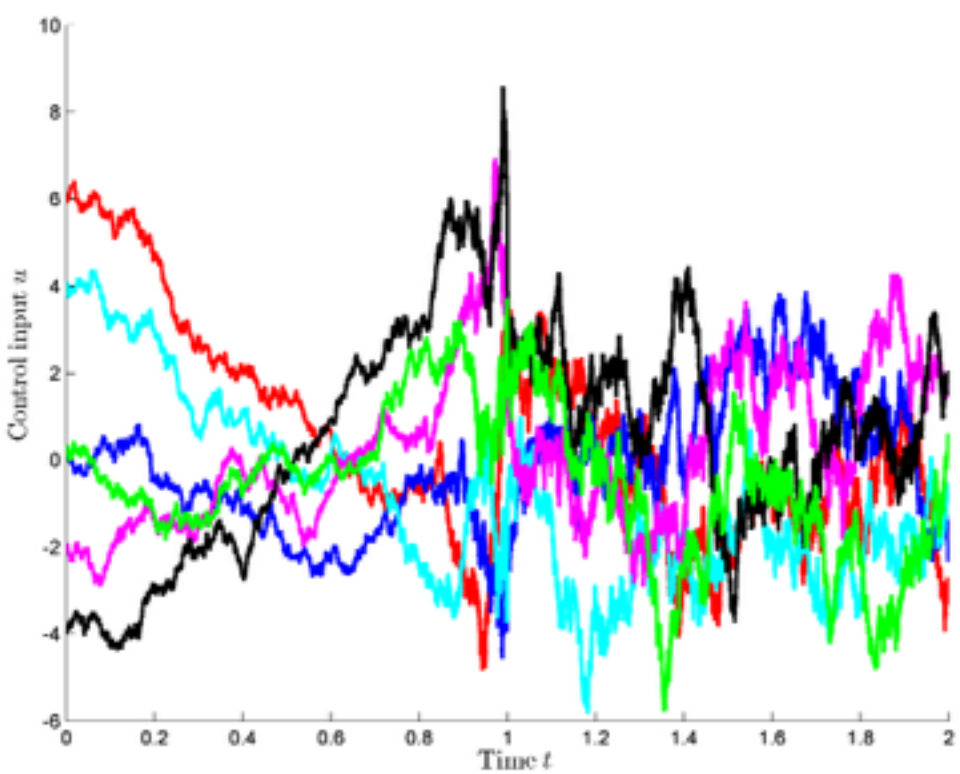
$$0 = (A - BK)\xi + Bu$$

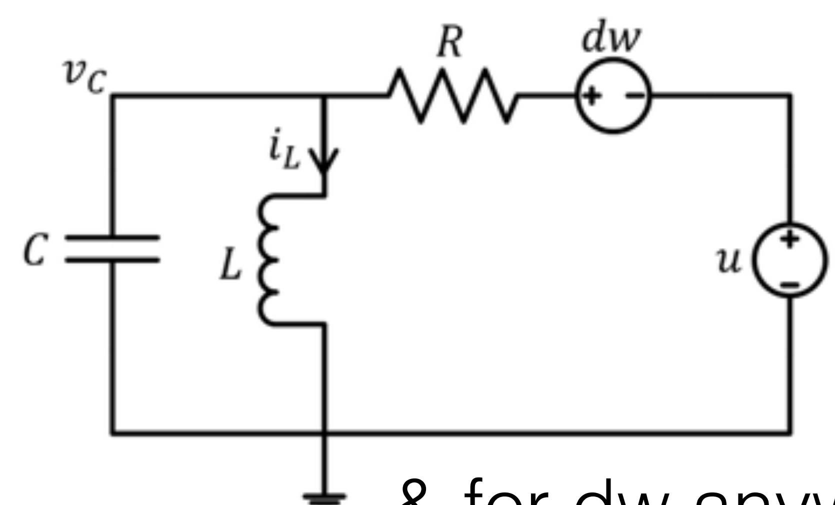
Schrödinger system

$$
\begin{aligned}
\dot{\Pi} &= -A'\Pi - \Pi A + \Pi BB'\Pi \\
\dot{\mathrm{H}} &= -A'\mathrm{H} - \mathrm{H}A - \mathrm{H}BB'\mathrm{H} \\
&\qquad + (\Pi + \mathrm{H})\,(BB' - B_1B_1')\,(\Pi + \mathrm{H}) \\
\Sigma_0^{-1} &= \Pi(0) + \mathrm{H}(0) \\
\Sigma_T^{-1} &= \Pi(T) + \mathrm{H}(T).
\end{aligned}
$$

Fast “cooling” + stationary control







& for dw anywhere

Open problem

Density matrices: e.g.
$\mathfrak{D} = \{\rho \geq 0 \mid \text{ symmetric } \rho \in \mathbb{R}^{n\times n} \text{ with trace}(\rho) = 1\}$

E_i with $i = 1, \ldots, n_{\mathcal{E}}$ and $\sum_{i=1}^{n_{\mathcal{E}}} E_i^\dagger E_i = I$
(typically $n_{\mathcal{E}} \sim n^2$
for "positivity-improving": $\rho \geq 0 \Rightarrow \mathcal{E}(\rho) > 0$)

TPTP: $\mathcal{E} \; : \; \mathfrak{D} \to \mathfrak{D} \; : \; \rho \longrightarrow \sigma = \sum_{i=1}^{n_{\mathcal{E}}} E_i \rho E_i^\dagger$

Data: ρ_0, ρ_T, $\mathcal{E}$.
Problem: Prove that the iteration:

$$\begin{aligned}
\mathcal{E} &: \hat{\phi}_0 \mapsto \hat{\phi}_T = \mathcal{E}(\hat{\phi}_0) \\
D_T &: \hat{\phi}_T \mapsto \phi_T = \left(\rho_T^{1/2}\left(\rho_T^{-1/2}\hat{\phi}^{-1}\rho_T^{-1/2}\right)^{1/2}\rho_T^{1/2}\right)^2 \\
\mathcal{E}^\dagger &: \phi_T \mapsto \phi_0 = \mathcal{E}^\dagger(\phi_T) \\
\hat{D}_0 &: \phi_0 \mapsto \hat{\phi}_0 = (\phi_0)^{1/2}\rho(\phi_0)^{1/2}
\end{aligned}$$

has an attractive fixed point.

Software for numerical experimentation

http://www.ece.umn.edu/~georgiou/papers/schrodinger_bridge/

Thank you for your attention

http://arxiv.org/abs/1405.6650
Positive contraction mappings for classical and quantum Schrodinger systems

http://arxiv.org/abs/1407.3421
Stochastic bridges of linear systems

http://arxiv.org/abs/1410.1605
Optimal steering of inertial particles diffusing anisotropically with losses

arxiv.org/abs/1408.2222
Optimal steering of a linear stochastic system to a final probability distribution

arxiv.org/abs/1410.3447
Optimal steering of a linear stochastic system to a final probability distribution, Part II