Primal and dual stability criteria applied to large scale interconnected systems

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Outline

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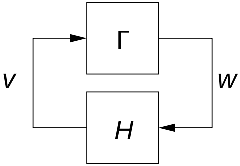
Primal and Dual Stability Criteria



Multiplier characterization

$$\Pi_\Gamma \subset \{\Pi \in \mathcal{S}_{\mathbf{C}}^{2n\times 2n} : \Gamma^*\Pi_{11}\Gamma + \Gamma^*\Pi_{12} + \Pi_{12}^*\Gamma + \Pi_{22} \leq 0\},$$

- Complete characterization is usually difficult to find or use.
- Convexifying assumptions convenient to use.

(*a*) **Primal condition:** For every $\omega \in [0, \infty]$ there exists $\Pi \in \Pi_{\Gamma(j\omega)}$ such that

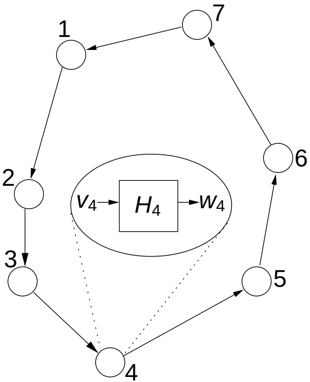
$$\begin{bmatrix} I \\ H(j\omega) \end{bmatrix}^* \Pi \begin{bmatrix} I \\ H(j\omega) \end{bmatrix} > 0 \tag{1}$$

(*b*) **Dual condition:** For every $\omega \in [0, \infty]$ we have

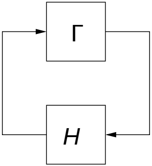
$$\begin{bmatrix} I \\ H(j\omega) \end{bmatrix} Z \begin{bmatrix} I \\ H(j\omega) \end{bmatrix}^* \notin \Pi^{\ominus}_{\Gamma(j\omega)} \tag{2}$$

for all $Z \in \mathcal{S}^{n\times n}_{\mathbf{C}}$ such that $Z \geq 0$ and $\mathrm{tr}(Z) = 1$.

Large Scale Interconnected Systems



$\Longleftrightarrow$



- H: $\begin{bmatrix} H_1 & & \\ & \ddots & \\ & & H_n \end{bmatrix}$
- Γ: interconnection operator.

- Explore structure to reduce computational complexity.
- Graphical interpretation of stability criterion.

References

Some references behind the talk

Author	Year	Journal
U. Jönsson and C.-Y. Kao	2008	MTNS08, ACC09, IEEE TAC
U. Jönsson and C.-Y. Kao	2010	T.B.S SICE conf. 2010
U. Jönsson	2010	ACC 2010

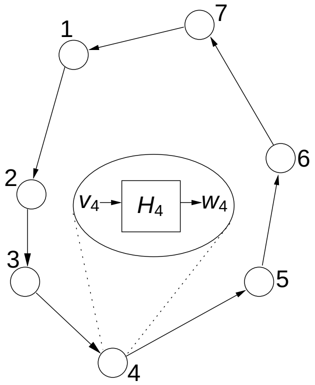
Some related works

Author	Year	Journal
J. A. Fax & R. M. Murray	2004	IEEE TAC
G. Vinnicombe	2002	Submitted
I. Lestas & G. Vinnicombe	2005	IFAC 2005
I. Lestas & G. Vinnicombe	2006	IEEE TAC

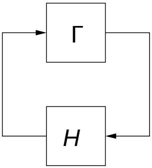
Outline

1. Introduction
2. Large Scale Interconnected Systems
3. Applications
4. Summary

Large Scale Interconnected Systems



$\Longleftrightarrow$



- H: $\begin{bmatrix} H_1 & & \\ & \ddots & \\ & & H_n \end{bmatrix}$
- Γ: interconnection operator.

- Focus on the case when Γ is normal.

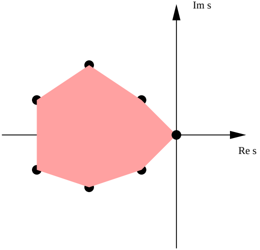
Identical Multipliers for all Subsystems

$$\Pi_{\Gamma,id} = \left\{ \begin{bmatrix} \pi_{11} I_n & \pi_{12} I_n \\ \pi_{12}^* I_n & \pi_{22} I_n \end{bmatrix} \in \mathcal{S}_{\mathbf{C}}^{2n\times 2n} : \Gamma^*\Gamma\pi_{11} + \Gamma^*\pi_{12} + \bar{\pi}_{12}\Gamma + \pi_{22} \leq 0 \right\}$$

$$\Pi_{\Gamma,id}^{\ominus} = \left\{ \begin{bmatrix} W_{11} & W_{12} \\ W_{12}^* & W_{22} \end{bmatrix} \in \mathcal{S}_{\mathbf{C}}^{2n\times 2n} : \begin{bmatrix} \mathrm{tr}(W_{11}) & \mathrm{tr}(W_{12}) \\ \mathrm{tr}(W_{12}^*) & \mathrm{tr}(W_{22}) \end{bmatrix} \in \mathrm{cone} \left\{ \begin{bmatrix} |\lambda_k|^2 & \lambda_k \\ \bar{\lambda}_k & 1 \end{bmatrix} : \lambda_k \in \mathrm{eig}(\Gamma) \right\} \right\}$$

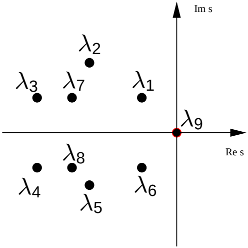
Descriptive power of the multipliers

$$\Pi_{\Gamma,id} = \left\{ \begin{bmatrix} \pi_{11} I_n & \pi_{12} I_n \\ \pi_{12}^* I_n & \pi_{22} I_n \end{bmatrix} \in \mathcal{S}_{\mathbf{C}}^{2n\times 2n} : \Gamma^*\Gamma\pi_{11} + \Gamma^*\pi_{12} + \bar{\pi}_{12}\Gamma + \pi_{22} \le 0 \right\}$$



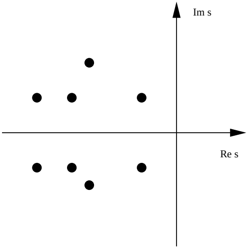
$\pi_{11} \ge 0 \quad \pi_{22} \le 0$

Robustness



$\pi_{22} \le 0$

Less conservative



Exact eigenvalue description

Requires homotopy

Robust Nyquist Criterion ($\pi_{11} \geq 0$ and $\pi_{22} \leq 0$)

Proposition

The feedback interconnection is stable if

(*i*) $\text{eig}(\Gamma(j\omega)) \in \Lambda := \text{co}\{\lambda_1, \ldots, \lambda_n\}$, $\forall \omega \in \mathbf{R} \cup \{\infty\}$,

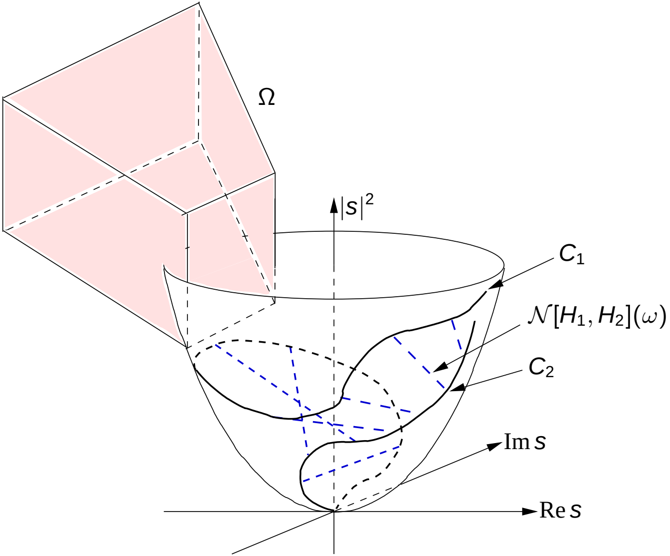
(*ii*) $\mathcal{N}[H_1, \ldots, H_n](\omega) \cap \Omega = \emptyset$, $\forall \omega \in \mathbf{R} \cup \{\infty\}$,

where the *extended Nyquist polytope* is defined as

$$\mathcal{N}[H_1, \ldots, H_n](\omega) = \text{co}\{(\text{Re}\, H_k(j\omega), \text{Im}\, H_k(j\omega), |H_k(j\omega)|^2) : k = 1, \ldots, n\}$$

and the *instability region*

$$\Omega = \left\{\alpha \cdot \text{co}\left\{\left(\text{Re}\,\frac{1}{\lambda_k}, \text{Im}\,\frac{1}{\lambda_k}, \frac{1}{|\lambda_k|^2}\right) : \lambda_k \neq 0\right\} + (0, 0, \theta_{n+1}) : \alpha \geq 1, \theta_{n+1} \geq 0\right\}$$


Ω
$|s|^2$
C_1
$\mathcal{N}[H_1, H_2](\omega)$
C_2
Im s
Re s

Generalized Nyquist Criteria ($\pi_{22} \leq 0$)

Proposition

The feedback interconnection is stable if

(i) $\mathrm{eig}(\Gamma(j\omega)) = \{\lambda_1, \ldots, \lambda_n\}$, $\forall \omega \in \mathbf{R} \cup \{\infty\}$,

(ii) $\mathcal{N}[H_1, \ldots, H_n](\omega) \cap \Omega_e = \emptyset$, $\forall \omega \in \mathbf{R} \cup \{\infty\}$,

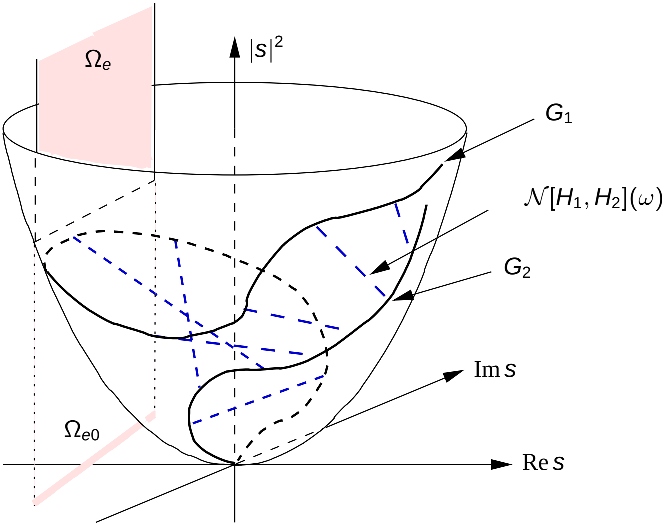
(iii) $\exists H_0 \in \mathcal{A}$ such that $H_0(j\omega) \in \mathrm{co}\{H_1(j\omega), \ldots, H_n(j\omega)\}$, $\forall \omega \in \mathbf{R} \cup \{\infty\}$ and $(I - H_0\Gamma)^{-1} \in \mathcal{A}^{n \times n}$,

where the *extended Nyquist polytope* is defined as

$$\mathcal{N}[H_1, \ldots, H_n](\omega) = \mathrm{co}\{(\mathrm{Re}\, H_k(j\omega), \mathrm{Im}\, H_k(j\omega), |H_k(j\omega)|^2) : k = 1, \ldots, n\}$$

and where the *instability region* is

$$\Omega_e = \mathrm{co}\left\{\left(\mathrm{Re}\,\frac{1}{\lambda_k}, \mathrm{Im}\,\frac{1}{\lambda_k}, \frac{1}{|\lambda_k|^2}\right) : \lambda_k \neq 0\right\} + (0, 0, \mathbf{R}_+).$$



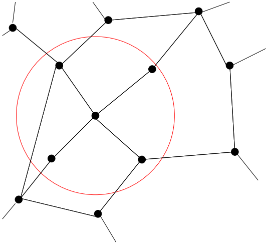
$|s|^2$
Ω_e
G_1
$\mathcal{N}[H_1, H_2](\omega)$
G_2
$\mathrm{Im}\, s$
Ω_{e0}
$\mathrm{Re}\, s$

Comments

- The primal criterion has an equivalent time-domain formulation
  - Applies to time-varying and nonlinear systems.
  - Looses the duals graphical interpretation.

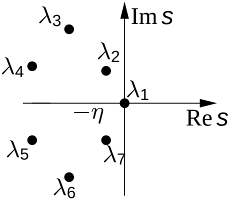
Outline

Application I: Consensus Networks



$$\dot{y}_k = p_k * \sum_{l \in \mathcal{N}_k} (y_l - y_k), \quad k = 1, \ldots, N$$

- $\mathcal{N}_k$ denotes the set of neighbors of node k.
- p_k is the convolution kernel corresponding to an stable linear system $P_k \in \mathcal{A}$ with $P_k(0) > 0$.



Proposition

Suppose

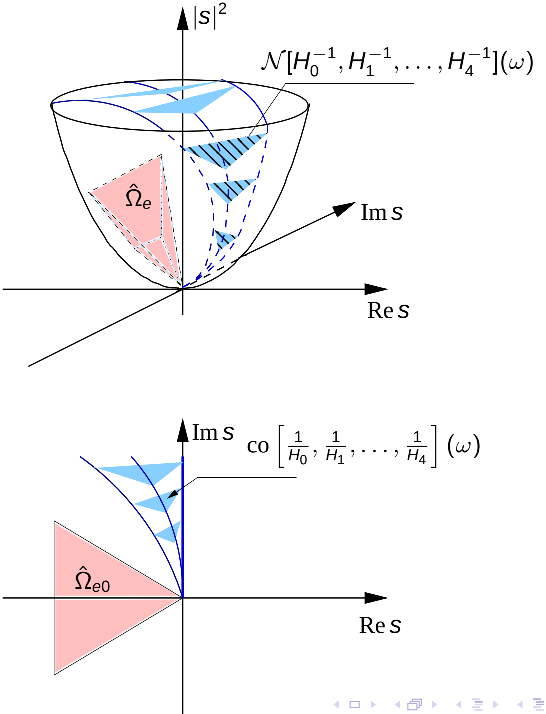
(*i*) $\mathrm{eig}(\Gamma) = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$, where

$$\mathrm{Re}\,\lambda_n \leq \mathrm{Re}\,\lambda_{n-1} \leq \dots \leq \mathrm{Re}\,\lambda_2 < \lambda_1 = 0$$

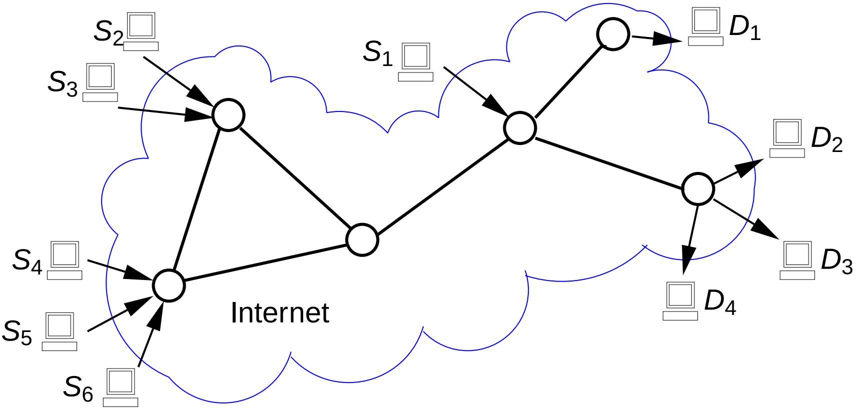
(*ii*) $\mathcal{N}[H_0^{-1}, H_1^{-1}, \dots, H_n^{-1}](\omega) \cap \mathrm{co}\{(\lambda_k, |\lambda_k|^2), k = 1, \dots, n\} = \emptyset$, $\forall \omega$, where

$$H_0(s) = \frac{1}{s} \qquad H_k(s) = \frac{1}{s} P_k(s)$$

Then $y(t) \to \mathrm{span}\{\mathbf{1}\}$.

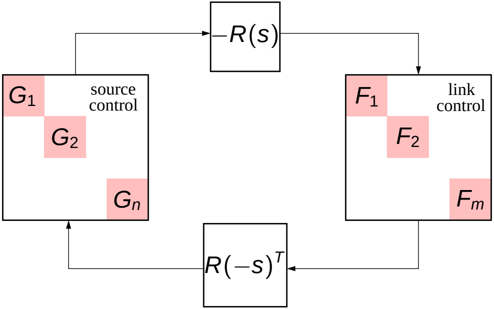

$|s|^2$
$\mathcal{N}[H_0^{-1}, H_1^{-1}, \ldots, H_4^{-1}](\omega)$
$\hat{\Omega}_e$
Im s
Re s
Im s
$\mathrm{co}\left[\frac{1}{H_0}, \frac{1}{H_1}, \ldots, \frac{1}{H_4}\right](\omega)$
$\hat{\Omega}_{e0}$
Re s

Application II: Internet Congestion Control



The General Case

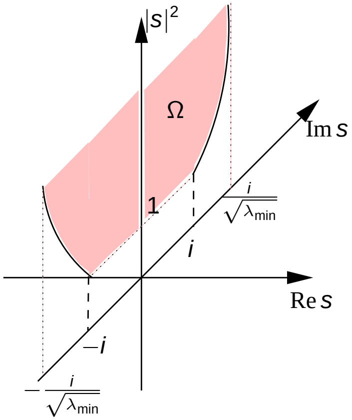
Dynamics around the equilibrium point.



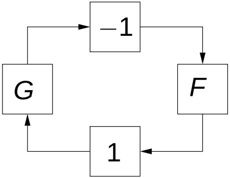
$$H(s) = \begin{bmatrix} G(s) & 0 \\ 0 & F(s) \end{bmatrix}, \qquad \Gamma(s) = \begin{bmatrix} 0 & R(-s)^T \\ -R(s) & 0 \end{bmatrix}$$

Identical Multipliers for all Subsystems

$$\Pi_\Gamma = \left\{ \begin{bmatrix} \pi_{11} I_n & \pi_{12} I_n \\ \pi_{12}^* I_n & \pi_{22} I_n \end{bmatrix} : \Gamma^*\Gamma\pi_{11} + \Gamma^*\pi_{12} + \pi_{12}^*\Gamma + \pi_{22} \leq 0;\ \pi_{22} \leq 0 \right\},$$



Exploring the Bi-Partite Structure Using Multipliers



$$H(s) = \begin{bmatrix} G(s) & 0 \\ 0 & F(s) \end{bmatrix}, \qquad \Gamma(s) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

The multipliers

$$\Pi_\Gamma = \left\{ \begin{bmatrix} x_1 & & y & \\ & x_2 & & \bar{y} \\ \bar{y} & & -x_2 & \\ & y & & -x_1 \end{bmatrix} : x_1, x_2 \geq 0; y \in \mathbf{C} \right\}$$

leads to a dual with the frequency domain interpretation

$$G(j\omega)K(j\omega) \notin (-\infty, -1]$$

Outline

Summary

- Primal stability condition
  - Multiplier-based criterion.
  - Time-domain interpretation generalizes to LTV and nonlinear systems.
- Dual stability condition
  - Robust Nyquist-type criteria in three dimensions