Distributed Synthesis and Validation of Model Predictive Controllers

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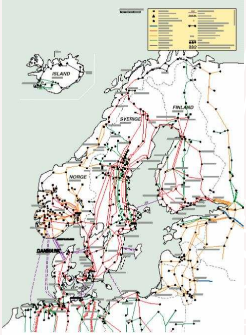
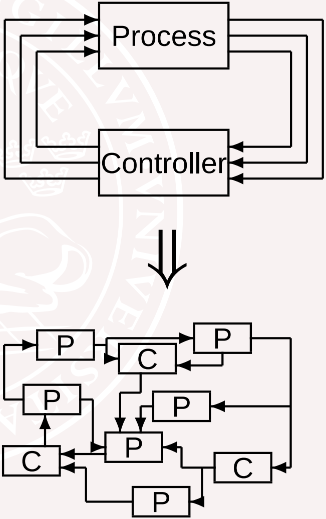
Building theoretical foundations for distributed control

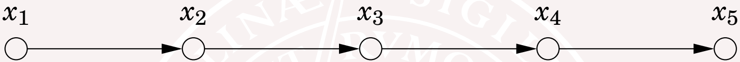




We need methodology for

- Decentralized specifications
- Decentralized design
- Verification of global behavior

Example 1: A vehicle formation

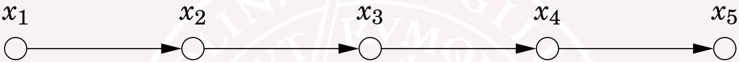


Each vehicle obeys the independent dynamics

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \\ x_3(t+1) \\ x_4(t+1) \end{bmatrix} = \begin{bmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} B_1 u_1(t) + w_1(t) \\ B_2 u_2(t) + w_2(t) \\ B_3 u_3(t) + w_3(t) \\ B_4 u_4(t) + w_4(t) \end{bmatrix}$$

The objective is to make $\mathbf{E}|Cx_{i+1} - Cx_i|^2$ small for $i = 1, \ldots, 4$.

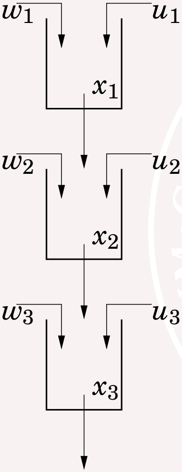
Example 2: A supply chain for fresh products



Fresh products degrade with time:

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \\ x_3(t+1) \\ x_4(t+1) \end{bmatrix} = \begin{bmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} -u_1(t) + w_1(t) \\ u_1(t) - u_2(t) \\ u_2(t) - u_3(t) \\ u_3(t) + w_4(t) \end{bmatrix}$$

Example 3: Water distribution systems



$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \\ x_3(t+1) \\ x_4(t+1) \end{bmatrix} = \begin{bmatrix} * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ 0 & * & * & 0 \\ 0 & 0 & * & * \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} B_1u_1 + w_1 \\ B_2u_2 + w_2 \\ B_3u_3 + w_3 \\ B_4u_4 + w_4 \end{bmatrix}$$

Example 4: Wind farms

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \\ x_3(t+1) \\ x_4(t+1) \end{bmatrix} = \begin{bmatrix} * & * & 0 & 0 \\ * & * & * & 0 \\ 0 & * & * & * \\ 0 & 0 & * & * \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} B_1u_1(t) + w_1(t) \\ B_2u_2(t) + w_2(t) \\ B_3u_3(t) + w_3(t) \\ B_4u_4(t) + w_4(t) \end{bmatrix}$$

The objective is to make $\sum_i \mathbf{E}|x_i|^2$ small. Can we get a solution where turbines only communicate with neighbors?

A long history

The saddle algorithm:
Arrow, Hurwicz, Usawa 1958

Books on control and coordination in hierarchical systems:
Mesarovic, Macko, Takahara 1970
Singh, Titli 1978
Findeisen 1980

Major application to water supply network:
Carpentier and Cohen, Automatica 1993

Our contribution:
Analyze effects of suboptimality

Outline

- Introduction
- **Dynamic dual decomposition**
- Distributed Model Predictive Control

A control problem with graph structure

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \\ \vdots \\ x_{\mathcal{J}}(t+1) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & & 0 \\ A_{21} & \ddots & \ddots & \\ & \ddots & \ddots & A_{(\mathcal{J}-1)\mathcal{J}} \\ 0 & & A_{\mathcal{J}(\mathcal{J}-1)} & A_{\mathcal{J}\mathcal{J}} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{\mathcal{J}}(t) \end{bmatrix} + \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_{\mathcal{J}}(t) \end{bmatrix}$$

Minimize $V = \sum_{i=1}^{\mathcal{J}} \sum_{t=0}^{N} \left(|x_i(t)|^2 + |u_i(t)|^2 \right)$

subject to $x(0) = x^0$ while $x_i(t) \in X_i$ and $u_i(t) \in U_i$ for all i, t.

(Assume $X_1 \times X_2 \times \cdots \times X_{\mathcal{J}}$ control invariant.)

Decomposing the problem

Minimize $V = \sum_{i=1}^{\mathcal{J}} \sum_{t=0}^{N} \left(|x_i(t)|^2_{Q_i} + |u_i(t)|^2_{R_i} \right)$

subject to

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \\ \vdots \\ x_{\mathcal{J}}(t+1) \end{bmatrix} = \begin{bmatrix} A_{11}x_1(t) \\ A_{22}x_2(t) \\ \vdots \\ A_{\mathcal{J}\mathcal{J}}x_{\mathcal{J}}(t) \end{bmatrix} + \begin{bmatrix} v_1(t) \\ v_2(t) \\ \vdots \\ v_{\mathcal{J}}(t) \end{bmatrix} + \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_{\mathcal{J}}(t) \end{bmatrix}$$

where $x(0) = x^0$ and

$$v_i(t) = \sum_{j \neq i} A_{ij} x_j(t) \qquad \text{for all } i, t$$

Decomposing the Cost Function

$$\max_p \min_{u,v,x} \sum_{t=0}^{N} \sum_{i=1}^{\mathcal{J}} \left[|x_i|^2_{Q_i} + |u_i|^2_{R_i} + 2p_i^T \left(v_i - \sum_{j \neq i} A_{ij} x_j \right) \right]$$

$$= \max_p \sum_i \min_{u_i, v_i, x_i} \sum_{t=0}^{N} \left[|x_i|^2_{Q_i} + |u_i|^2_{R_i} + 2p_i^T v_i - 2x_i^T \left(\sum_{j \neq i} A_{ji}^T p_j \right) \right]$$

so, given the sequences $\{p_j(t)\}_{t=0}^N$, agent i should minimize

$$\underbrace{\sum_{t=0}^{N} |x_i|^2_{Q_i} + |u_i|^2_{R_i}}_{\text{local cost}} + \overbrace{2\sum_{t=0}^{N} p_i^T v_i}^{\text{what he expects others to charge him}} - \underbrace{2\sum_{t=0}^{N} x_i^T \left(\sum_{j \neq i} A_{ji}^T p_j \right)}_{\text{what he is payed by others}}$$

with $x_i(t+1) = A_{ii} x_i(t) + v_i(t) + u_i(t)$ and $x_i(0), v_i(0)$ fixed.

Distributed Optimization Procedure

Local optimizations in each node

$$V_i^{N,p}(\bar{x}_i) = \min_{u_i, v_i, x_i} \sum_{\tau=0}^{N} \ell_i^p\big(x_i(\tau), u_i(\tau), v_i(\tau)\big)$$

can be coordinated by (local) gradient updates of the prices

$$p_i^{k+1}(\tau) = p_i^k(\tau) + \gamma_i^k \left[v_i^k(\tau) - \textstyle\sum_{j \neq i} A_{ij} x_j^k(\tau) \right]$$

Future prices included in negotiation for first control input!

Finite horizon error bounds available under different types of assumptions on the step size sequence γ_i^k.

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"Wind Farm" Revisited

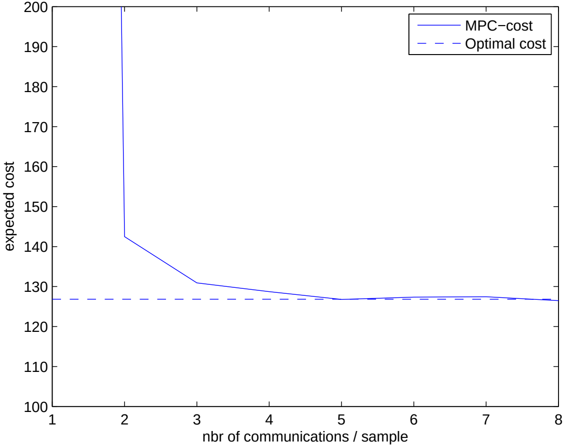
Minimize $V = \mathbf{E} \sum_{i=1}^{n} \left(|x_i|^2 + |u_i|^2\right)$ subject to

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \\ \vdots \\ x_n(t+1) \end{bmatrix} = \begin{bmatrix} 0.6 & 0.1 & & 0 \\ 0.3 & \ddots & \ddots & \\ & \ddots & \ddots & 0.1 \\ 0 & & 0.3 & 0.6 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} u_1(t) + w_1(t) \\ u_2(t) + w_2(t) \\ \vdots \\ u_n(t) + w_n(t) \end{bmatrix}$$

We will solve this by "distributed MPC". For every t, the agents measure their local state $x_i(t)$. The vector of future prices is then updated by K gradient iterations starting from the prices computed at $t-1$ for a time horizon of length N.

Re-negotiation of future prices at every time step!

Performance Versus Number of Gradient Iterations



A distributed controller with 100 agents, using only local data.
Performance close to optimal centralized controller!

Challenges for theory

- What prediction horizon is needed?
- How many gradient iterations for the prices?

References:
Grüne and Rantzer, IEEE TAC October 2008.
Pannek, PhD thesis 2009
Giselsson and Rantzer, submission for CDC 2010.

Definition of Model Predictive Controller

With

$$V_i^{N,p}(x_i^0) = \min_u \sum_{t=0}^{N} l_i^p(x_i(t), u_i(t))$$

the corresponding model predicitve controller $u_i = \mu_i^{N,p}(x)$ gives the inifinite horizon cost

$$\sum_{t=0}^{\infty} l(x(t), \mu^{N,p}(x(t))$$

where $x(t+1) = f(x(t), \mu^{N,p}(x(t))), \quad x(0) = x^0$

How does this compare to the optimal cost $V_\infty(x^0)$?

Distributed Validation of MPC Accuracy

Suppose that $\beta \in [0, 1]$ and

$$V_i^{N,p}(x_i(t)) \geq V_i^{N,p}(x_i(t+1)) + \beta l_i^p(x_i, \mu_i^{N,p}(x))$$

for all x along a trajectory obtained by starting in x^0 and applying the control law $\mu^{N,p}$. Then

$$\beta \sum_{t=0}^{\infty} l(x(t), \mu^{N,p}(x(t)) \leq V_\infty(x^0)$$

Hence the distance to optimality can be rigorously bounded from trajectory data!

Conclusions on Distributed MPC

We have synthesized a game that solves optimal control problems via independent decision-makers in every node, acting in their own interest!

- Optimal strategies independent of global graph structure!
- States are measured only locally
- Linearly complexity (given horizon and iteration scheme)
- Distributed bounds on distance to optimality