Multiplexed Model Predictive Control

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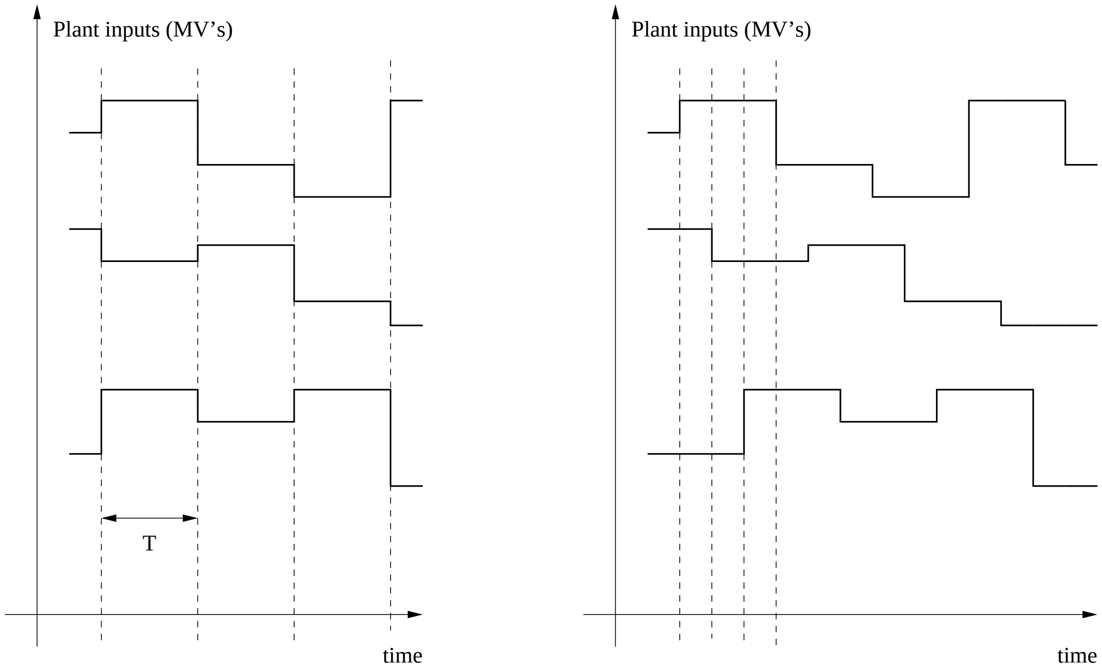
Multiplexed Model Predictive Control (MMPC)

- MPC increasingly used for fast systems.
- Reduce complexity of optimization problem.
- Various distributed schemes proposed
  — but all with synchronous control updates.
- MMPC updates 1 input at a time (or subset of inputs).
- *Do something now* better than *Do optimal thing later*?
- Motivation: Speed, Scaling with number of inputs.

Multiplexed MPC — multi-input systems

Sequential updates of control inputs



Left: Conventional MPC, Right: Multiplexed MPC

Assumptions

- Plant has m inputs. Update cycle period is T.
- Only one input updated at each time step (at time kT/m), in sequence.
- Optimise only one input over future horizon at each step. *Can be thought of as m controllers.*
- Measurements of state vector are made at intervals of T/m.
- Current state x_k is known when deciding the update of each input. *x_k is known to each controller.*
- $N = (N_u - 1)m + 1$, where N_u is the number of moves to be optimized *per input channel.*

Update one input at a time

Plant: If only one input is updated at each k then

$$\begin{aligned} x_{k+1} &= Ax_k + \sum_{j=1}^{m} B_j u_{j,k} \\ &= Ax_k + B_{\sigma(k)} \tilde{u}_k \end{aligned}$$

where

$$\sigma(k) = (k \mod m) + 1$$

is a **periodic switching function**: $\sigma(k+m) = \sigma(k)$

So multi-input LTI plant looks like **periodic single-input** plant.

Periodic Invariance

- Assume $\tilde{u}_{k+N-1} = -K_{\sigma(k)} x_{k+N-1}$ (beyond horizon).
- $K_{\sigma(k)}$ stabilises the periodic system.
- $\big(\mathcal{X}_I(K_{\sigma(k)})\big)$ is 'periodically invariant' sequence of sets:

$$x_k \in \mathcal{X}_I(K_{\sigma(k)}) \text{ and } -K_{\sigma(k)} x_k \in \mathbb{U}_{\sigma(k)} \Rightarrow \\ (A - B_{\sigma(k)} K_{\sigma(k)}) x_k \in \mathcal{X}_I(K_{\sigma(k+1)})$$

and

$$\mathcal{X}_I(K_{\sigma(k)}) \subset \mathbb{X},$$

for $\sigma(k) = 1, \ldots, m$.

What will the non-optimised inputs do?

- Controller j decides future sequence of j'th input only.
- Other inputs are treated as *known disturbances*.
- Assume that controller j knows the future plans of the other controllers, and assumes $u_{\sigma(k),k+i} = -K_{\sigma(k)} x_{k+i}$ beyond the planning horizon.

What will the optimised inputs do?

- **Either** remain fixed until the next optimisation,
- **Or** varies as $u_{\sigma(k),k+i} = -K_{\sigma(k)} x_{k+i}$.

Optimisation problem: Basic MMPC

$$
\begin{aligned}
\text{Minimise} \quad & J_k = F_{\sigma(k)}(x_{k+N|k}) + \textstyle\sum_{i=0}^{N-1} \left(\|x_{k+i|k}\|_q^2 + \|\tilde{u}_{k+i|k}\|_r^2\right) \\
\text{wrt} \quad & u_{k+i|k}, \quad (i = 0, m, 2m, \ldots, N-1) \\
\text{s.t.} \quad & u_{k+i|k} \in \mathbb{U}_{\sigma(k+i)}, \quad (i = 0, \ldots, N-1) \\
& x_{k+i|k} \in \mathbb{X}, \quad (i = 1, \ldots, N) \\
& x_{k+N|k} \in \mathcal{X}_I(K_{\sigma(k)}) \\
& -K_{\sigma(k+N)} x_{k+N|k} \in \mathbb{U}_{\sigma(k+N)} \\
& x_{k+i+1|k} = A x_{k+i|k} + B_{\sigma(k+i)} u_{k+i|k} \\
& u_{k+i|k} = u_{k+i|k-1}, \quad (i \neq 0, m, \ldots, N-1).
\end{aligned}
$$

Note: $F_{\sigma(k)}(x_{k+N|k}) \geq 0$ is a *terminal cost.*

Basic MMPC Algorithm

1. Initialise by solving the optimisation problem, but *optimising over all the variables* $\tilde{u}_{k+i|k}, i = 0, 1, \ldots, N-1$.
2. Apply control move $u_{\sigma(k),k} = \tilde{u}_{k|k}$
3. Store planned moves $\vec{u}_{k,m|k}$.
4. Pause for one time step, increment k, obtain new measurement x_k.
5. Solve the optimisation problem.
6. Go to step 2.

Implicit assumption: N is large enough.

Stability Theorem

MMPC, obtained by implementing the basic MMPC Algorithm, gives closed-loop stability if the problems are well-posed, and if the set of terminal costs $\{F_\sigma(\cdot)\}$ satisfies

$$F_{\sigma^+}([A - B_{\sigma^+} K_{\sigma^+}]x) + \|x\|_q^2 + \|K_{\sigma^+} x\|_r^2 \le F_\sigma(x) \quad \text{for} \quad \sigma = 1, \ldots, m.$$

where $\sigma^+ = (\sigma \bmod m) + 1$, namely the cyclical successor value to σ.

Standard MPC proof, using the value function as a Lyapunov function.

Robust MMPC

Suppose that

- $$x_{k+1} = Ax_k + \sum_{j=1}^{m} B_j \Delta u_{j,k} + Ew_k.$$
- w_k satisifies $w_k \in \mathcal{W}\ \forall k$ and $\mathcal{W}$ is a known, bounded set containing 0.
- w_k is *not* measured — but can be estimated with 1-step delay.

Correct for disturbances, Tighten constraints

- $\tilde{u}_{k+i|k+1} = \tilde{u}^*_{k+N-1|k} + M_{i-1,\sigma(k+1)}Ew_k$ for $i = 1, \ldots, N-1$,
- $\mathcal{X}_{i+1,\sigma(k)} = \mathcal{X}_{i,\sigma(k+1)} \sim L_{i,\sigma(k+1)}E\mathcal{W}$ with $\mathcal{X}_{0,\sigma(k)} = \mathbb{X}$,
- $\mathcal{U}_{i,\sigma(k)} = \mathcal{U}_{i-1,\sigma(k+1)} \sim M_{i-1,\sigma(k+1)}E\mathcal{W}$ with $\mathcal{U}_{0,\sigma(k)} = \mathbb{U}_{\sigma(k)}$,
- $L_{i+1,\sigma(k)} = AL_{i,\sigma(k)} + B_{\sigma(k+i)}M_{i,\sigma(k)}$ with $L_{0,\sigma(k)} = I$.
- $M_{i,\sigma(k)}$ and $K_{\sigma(k)}$ are chosen off-line.
- Terminal sets have robust invariance property,
- Modify the constraints in the optimisation problem.

Robust MMPC Theorem

If the system is controlled using the Robust MMPC algorithm and the initial optimisation at time $k = k_0$ is feasible, and $x_{k_0} \in \mathbb{X}$, then:

1. the optimisation remains feasible, and
2. the constraints $x_k \in \mathbb{X}$ and $\tilde{u}_k \in \mathbb{U}_k$ are satisfied for $k > k_0$ and for all admissible disturbances.

Note:

- Feasibility $\Rightarrow$ stability.
- Tightening constraints reduces the chances of initial feasibility.

Applications and Examples

- μ-PCR temperature control (2^n regions). Interactions will increase and time constants decrease as n increases.
- A-7A Corsair II longitudinal flight dynamics, with 2 inputs and input disturbances. 25% better performance, $4\times$ speedup.
- Air-traffic management, with stochastic wind.

Air-traffic: En-route conflict resolution

- Coupling through constraints only — separation rules.
- Non-convex constraints. Stochastic wind disturbances.
- SESAR 'A3' concept: No ground ATC assistance.
- Assume System-Wide Information Management (SWIM) available.
- Treat each aircraft (agent) as an 'input'.
- Initial solution available from *Reference Business Trajectory*.
- Non-quadratic costs, finite duration (cf. Richards and How). Theorem still holds, gives guaranteed completion.
- Effective solutions for up to 6 aircraft, using CPLEX.
- Obvious protocol in case of comms failure.

Conclusions

- Multiplexed MPC updates one input at a time.
- *Do something sooner* can be better than *Do optimal thing later*.
- Basic and robust versions.
- Theoretical guarantees available.
- **Generalisations:** Unequal intervals; Groups of inputs; 'Channel-hopping' MPC.