Stabilization of Linear Systems Over Gaussian Relay Networks

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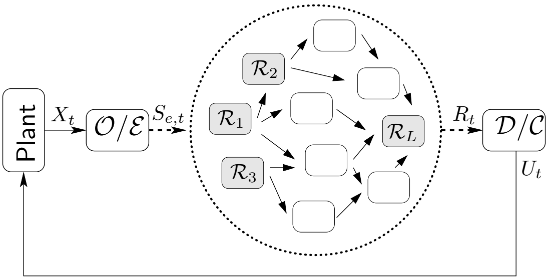
Closed-loop System

A discrete-time LTI system

$$X_{t+1} = AX_t + U_t + W_t$$

- The initial state X_0 has an arbitrary pdf with a given covariance Λ_0 ($\text{Tr}[\Lambda_0] < \infty$, $h(X_0) < \infty$)
- The process noise W_t is zero-mean i.i.d. Gaussian with covariance K_W ($\text{Tr}[K_W] < \infty$)
- The system matrix A has eigenvalues $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$, with $1 \leq |\lambda_i| < \infty$; i.e., the open loop system is unstable
- No constraint on the control signal U_t, in the general case

Stabilization over a Gaussian Relay Network



- Decoder/Controller policy

$$U_t = \pi_t(R_{[0,t]}), \qquad \text{with } R_{[0,t]} \triangleq \{R_0, R_1, \ldots, R_t\}$$

- Observer/Encoder policy

$$S_{e,t} = f_t(X_{[0,t]}), \quad E[S_{e,t}^2] \leq P_S$$

- Links are modeled as white Gaussian channels.

Closed-loop System

Objective: Find conditions on the matrix A such that the system can be mean-square stabilized.

Definition: Mean-square Stability

A system is said to be *mean-square stable* if there exists a constant $M < \infty$ such that $E[\|X_t\|^2] < M$ for all t.

Control over noisy Gaussian channels...

R. Bansal and T. Başar
Simultaneous design of measurement and control strategies for stochastic systems with feedback. *Automatica*,1989.

S. Tatikonda, A. Sahai, and S. Mitter
Stochastic linear control over a communication channel. *IEEE Trans. Automat. Control*, 2004.

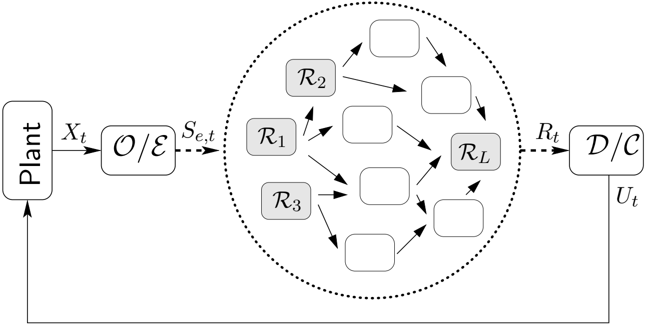
N. Elia
When Bode meets Shannon: Control-oriented feedback communication schemes. *IEEE Trans. Autom. Control*, 2004.

J. Freudenberg, R. Middleton, and V. Solo
Stabilization and disturbance attenuation over a Gaussian communication channel. *IEEE Trans. Automat. Control*, 2010.

S. Yüksel and T. Başar
Control over noisy forward and reverse channels. *IEEE Trans. Automat. Control*, 2011.

G. Lipsa and N. C. Martins
Optimal memoryless control of a delay in Gaussian noise: A simple counterexample. *Automatica*, 2011.

A Necessary Condition for Stabilization



Theorem

If the linear system can be mean-square stabilized over the Gaussian relay network, then

$$\log(|A|) \leq \liminf_{T\to\infty} \frac{1}{T} I\left(S_{e,[0,T-1]} \to R_{[0,T-1]}\right)$$

Proof

From the definition of directed information, and using properties of the system

$$I\left(S_{e,[0,T\text{-}1]} \to R_{[0,T\text{-}1]}\right) = \sum_{t=0}^{T-1} I\left(S_{e,[0,t]}; R_t | R_{[0,t\text{-}1]}\right)$$

$$\geq \{\,\text{standard tricks}\,\} \geq \sum_{t=0}^{T-1} I\left(X_t; R_t | R_{[0,t\text{-}1]}\right)$$

$$= I\left(X_0; R_0\right) + \sum_{t=1}^{T-1} (h(AX_{t-1} + U_{t-1} + W_{t-1} | R_{[0,t-1]}) - h(X_t | R_{[0,t]}))$$

$$\geq I\left(X_0; R_0\right) + \sum_{t=1}^{T-1} (\log(|A|) + h(X_{t-1} | R_{[0,t-1]}) - h(X_t | R_{[0,t]}))$$

$$= h\left(X_0\right) + (T-1)\log\left(|A|\right) - h\left(X_{T\text{-}1} | R_{[0,T\text{-}1]}\right)$$

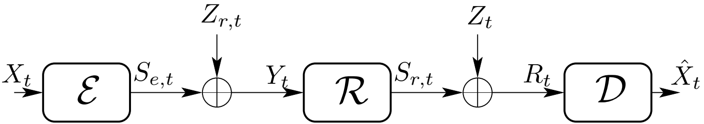
Hence, since $h(X_0) < \infty$,

$$\begin{aligned}
&\liminf_{T\to\infty} \frac{1}{T} I\left(S_{e,[0,T\text{-}1]} \to R_{[0,T\text{-}1]}\right) \\
&\qquad \geq \liminf_{T\to\infty} \frac{1}{T}\left(h\left(X_0\right) + (T-1)\log\left(|A|\right) - h\left(X_{T\text{-}1}|R_{[0,T\text{-}1]}\right)\right) \\
&\qquad = \log\left(|A|\right) - \limsup_{T\to\infty} \frac{1}{T} h\left(X_{T\text{-}1}|R_{[0,T\text{-}1]}\right)
\end{aligned}$$

When the system is stable $h\left(X_{T\text{-}1}|R_{[0,T\text{-}1]}\right) \leq h(X_{T\text{-}1}) < \infty$, thus

$$\liminf_{T\to\infty} \frac{1}{T} I\left(S_{e,[0,T\text{-}1]} \to R_{[0,T\text{-}1]}\right) \geq \log(|A|)$$

Example: Gaussian two-hop



$$E[S_{e,t}^2] \leq P_S,\ E[Z_{r,t}^2] = N_r,\ E[S_{r,t}^2] \leq P_R,\ E[Z_t^2] = N$$

$$\begin{aligned}
I\big(S_{e,[0,T-1]} \to R_{[0,T-1]}\big) \\
&\leq \min\left\{I\left(S_{e,[0,T-1]} \to Y_{[0,T-1]}\right), I\left(S_{r,[0,T-1]} \to R_{[0,T-1]}\right)\right\} \\
&\leq \min\left\{\sum_{t=0}^{T-1} I(S_{e,t};Y_t), \sum_{t=0}^{T-1} I(S_{r,t};R_t)\right\} \\
&\leq \frac{T}{2}\min\left\{\log\left(1+\frac{P_S}{N_r}\right), \log\left(1+\frac{P_R}{N}\right)\right\}
\end{aligned}$$

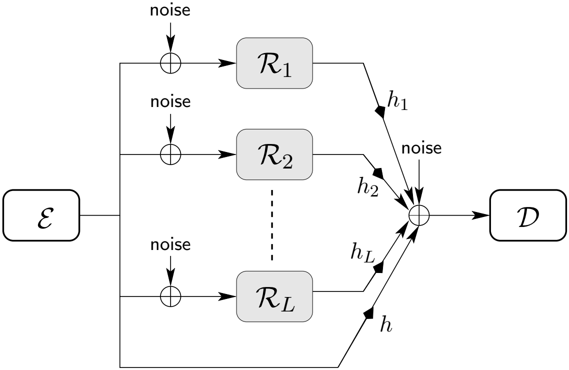
Hence, if the system can be stabilized

$$\log(|A|) \leq \frac{1}{2}\min\left\{\log\left(1+\frac{P_S}{N_r}\right), \log\left(1+\frac{P_R}{N}\right)\right\}$$

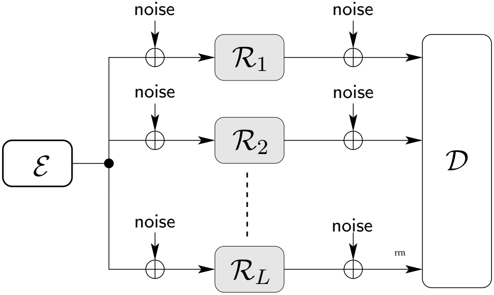
Sufficient Conditions, Different Topologies

We have derived a set of sufficient conditions for several different configurations:

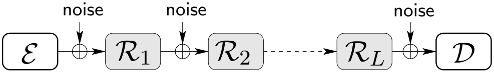
Non-orthogonal, half/full-duplex



Orthogonal, half-duplex

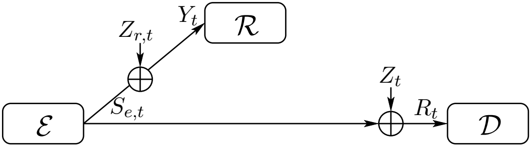


Cascade/Multi-hop

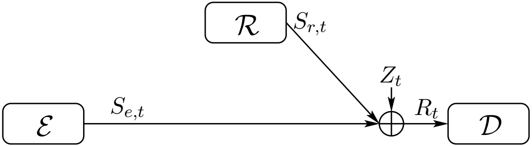


Half-duplex Gaussian Relay Channel

For simplicity, we illustrate the technique for a scalar system over a special case of the general non-orthogonal network:



(a) First transmission phase



(b) Second transmission phase

- Scalar system

$$X_{t+1} = \lambda X_t + U_t + W_t$$

$(E[X_0^2] = \Lambda_0,\ E[W_t^2] = K_W)$

- First ('odd') phase: $\mathcal{E}$ transmits with power $2\beta P_S$, where $0 < \beta \leq 1$
- Second ('even') phase: $\mathcal{E}$ and $\mathcal{R}$ transmit with powers $2(1-\beta)P_S$ and P_r, with $P_r \leq P_R$
- The destination receives

$$R_t = hS_{e,t} + Z_t \qquad t = 1, 3, 5, \ldots$$

$$R_t = hS_{e,t} + S_{r,t} + Z_t \qquad t = 2, 4, 6, \ldots$$

Proposition (sufficient condition)

The scalar system can be mean-square stabilized over the half-duplex Guassian relay channel if

$$\log(\lambda) < \frac{1}{4} \max_{\substack{0<\beta\leq 1 \\ 0\leq P_r \leq P_R}} \left(\log\left(1 + \frac{2h^2\beta P_S}{N}\right) + \log\left(1 + \frac{\tilde{M}(\beta, P_r)}{\tilde{N}(\beta, P_r)}\right)\right)$$

where $\tilde{N}(\beta, P_r) = \frac{P_r N_R}{2\beta P_S + N_R} + N$ and

$$\tilde{M}(\beta, P_r) = \left(\sqrt{2h^2(1-\beta)P_S} + \sqrt{\frac{2\beta P_S P_r N}{(2\beta P_S + N_R)(2h^2\beta P_S + N)}}\right)^2$$

Remarks:

- RHS turns out to be the directed information rate, *given that the system runs the protocol described in the proof*
- The condition does not depend on the process noise $\{W_t\}$

Proof Outline

1. Linear communication and control strategy
   - Initialization to obtain a Gaussian state
   - Odd and even phase transmission, inspired by Schalkwijk–Kailath coding
   - Amplify-and-forward relaying
   - Control based on MMSE estimation $\Rightarrow$ Gaussian state
   - Both signaling and control are *linear* operations

2. Derivation of the sufficient condition
   - Find a useful recursive representation for the second moment of the state process
   - Construct a majorizing sequence with a convergence criterion that provides a sufficient condition

Proof: Initialization

Initial time step, $t = 0$*:*

- $\mathcal{E}$ transmits $S_{e,0} = \sqrt{\frac{P_S}{\Lambda_0}} X_0$
- $\mathcal{R}$ neither receives nor transmits.
- $\mathcal{D}$ observes $R_0 = hS_{e,0} + Z_0$, and estimates

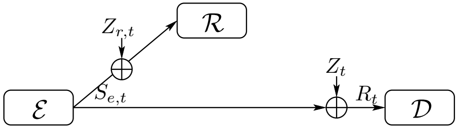
$$\hat{X}_0 = \frac{1}{h}\sqrt{\frac{\Lambda_0}{P_S}} R_0 = X_0 + \frac{1}{h}\sqrt{\frac{\Lambda_0}{P_S}} Z_0$$

- $\mathcal{C}$ takes an action $U_1 = -\lambda \hat{X}_0 \Rightarrow$

$$X_1 = \lambda(X_0 - \hat{X}_0) + W_0 = -\frac{\lambda}{h}\sqrt{\frac{\Lambda_0}{P_S}} Z_0 + W_0$$

$\Rightarrow$ new state is zero-mean Gaussian

Proof: 'Odd' Transmission Phase



$t = 1, 3, 5, ...$

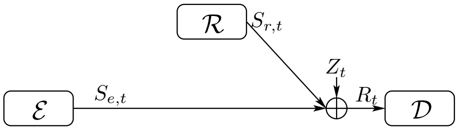
- $\mathcal{E}$ transmits $S_{e,t} = \sqrt{\frac{2\beta P_S}{\alpha_t}} X_t \quad (\alpha_t = E[X_t^2])$
- $\mathcal{R}$ receives but does not transmit
- $\mathcal{D}$ observes $R_t = hS_{e,t} + Z_t$, and computes

$$\hat{X}_t = E[X_t | R_1, R_2, ..., R_t] = E[X_t | R_t] = \frac{E[X_t R_t]}{E[R_t^2]} R_t$$

- $\mathcal{C}$ takes action $U_t = -\lambda \hat{X}_t \quad \Rightarrow X_{t+1} = \lambda(X_t - \hat{X}_t) + W_t$

$\Rightarrow$ new state is zero-mean Gaussian (& uncorrelated with $R_{[0,t]}$)

Proof: 'Even' Transmission Phase



$t = 2, 4, 6, \ldots$

- $\mathcal{E}$ transmits $S_{e,t} = \sqrt{\frac{2(1-\beta)P_S}{\alpha_t}} X_t$
- $\mathcal{R}$ transmits $S_{r,t} = \sqrt{\frac{P_r}{(2\beta P_S + N_R)}} (S_{e,t-1} + Z_{r,t-1})$
- $\mathcal{D}$ receives

$$R_t = hS_{e,t} + S_{r,t} + Z_t = \mathsf{const}_1 X_t + \mathsf{const}_2 X_{t-1} + \tilde{Z}_t,$$

computes $\hat{X}_t = E[X_t | R_1, ..., R_t]$ and takes action $U_t = -\lambda \hat{X}_t$
$\Rightarrow X_{t+1} = \lambda(X_t - \hat{X}_t) + W_t$

$\Rightarrow$ new state is zero-mean Gaussian & uncorrelated with $R_{[0,t]}$

Proof: Second Moment of State Process

With $\alpha_t = E[X_t^2]$ and after some work we get

$$\begin{aligned}
\alpha_t &= \lambda^2 \left(\tfrac{N}{2h^2\beta P_S + N}\right)\alpha_{t-1} + K_W, \qquad t = 2,4,6,\ldots \\
\alpha_t &= \lambda^2 \left(\lambda^2 k\, \alpha_{t-2} + K_W\right) f(\alpha_{t-2}) + K_W, \qquad t = 3,5,7,\ldots
\end{aligned}$$

where $f(\alpha_{t-2})$ is of the form

$$f(x) = \frac{a + \frac{b}{x}}{\left(c + \sqrt{d + \frac{b}{x}}\right)^2 + a + \frac{b}{x}}$$

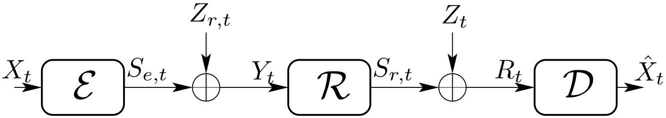
$\Rightarrow$ sufficient to look at odd time-instances; try to construct a convergent sequence $\{\alpha'_k\}$ which majorizes $\{\alpha_t\}_{t=2k+1}$

- ... possible, and α'_k is bounded if

$$\left(\frac{\lambda^4 k \tilde{N}(\beta, P_r)}{(k_2 + \sqrt{k_1 k})^2 + \tilde{N}(\beta, P_r)}\right) < 1$$

$\Rightarrow$ solving RHS for λ provides the stated sufficient condition

Special Case: Two-hop Relay Channel



(corresponds to $h = 0$ and $\beta = 1$)

Corollary

The scalar system can be mean-square stabilized over the two-hop relay channel if

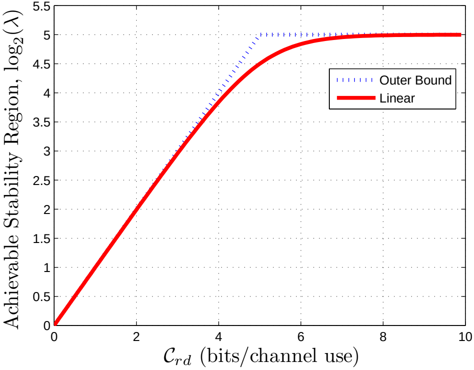
$$\log(\lambda) < \frac{1}{4} \log\left(1 + \frac{2P_S P_R}{P_R N_R + N(2P_S + N_R)}\right)$$

- The corresponding necessary condition is

$$\log(\lambda) < \frac{1}{4} \min\left\{\log\left(1 + \frac{2P_S}{N_R}\right), \log\left(1 + \frac{P_R}{N}\right)\right\}$$

$\Rightarrow$ *Necessary and sufficient coincide* if

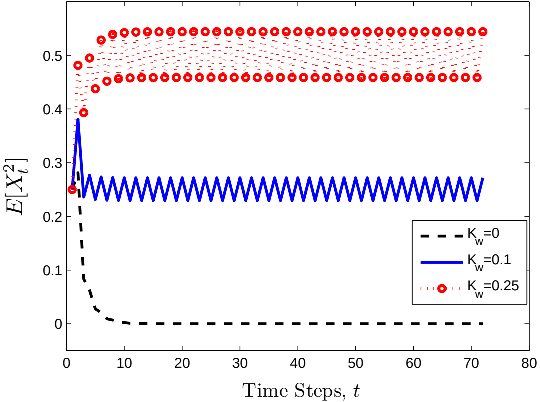
$$\frac{P_S/N_R}{P_R/N} \to \infty \quad \text{or} \quad \to 0$$



⇒ Linear policies *can be optimal*, however not in general, e.g.,

A. Zaidi, S. Yüksel, T. Oechtering and M. Skoglund,
On optimal policies for control and estimation over Gaussian relay channels, *Automatica* (submitted 2011, revised 2012)

Effect of Process Noise



- The stability condition does not change when the plant is noiseless, $K_W = 0$; however $E[X_t^2]$ is bounded away from zero if $K_W > 0$
- Stabilization at finite cost, $\lim_{T\to\infty} T^{-1} \sum_{t=0}^{T-1} E[U_t^2] < \infty$; also $K_W = 0 \Rightarrow E[U_t^2] \to 0$

Connection to Achievable Rates...

- As mentioned, the RHS in the criterion is equal to the directed information rate, when running the described protocol
- Consider instead the relay channel in isolation:
  - perfect feedback from $\mathcal{D}$ to $\mathcal{E}$
  - communication of a message in $\{1, 2, \cdots, K_n\}$
  - selected message W, decoded message $\hat{W}$
  - transmission over n uses of the channel
- Then, $R =$ the RHS is also an achievable rate, that is, there exists a coding scheme s.t.

$$\liminf_{n\to\infty} \frac{1}{n} \log K_n \geq R, \quad \lim_{n\to\infty} \mathsf{Pr}(\hat{W} \neq W) = 0$$

S. Bross and M. Wigger
On the relay channel with receiver–transmitter feedback.
IEEE Trans. Inform. Theory, 2009.

Multi-dimensional Systems. . .

Approach: Convey one component of the n-dimensional X_t at each time t

- For $m = 1, \ldots, n$, let

$$\gamma_m = \frac{\log(|\lambda_m|)}{\sum_{i=1}^{n} \log(|\lambda_i|)}$$

- Transmit the m-th component for a fraction γ_m of time
- Let ρ be the information rate corresponding to a scalar stabilization scheme (the RHS of the bound), then the m-th component can be stabilized if $\log(|\lambda_m|) < \gamma_m \rho$

$\Rightarrow$ the system will be stabilized if $\sum_m \log(|\lambda_m|) < \rho$

A. Zaidi, T. Oechtering, S. Yüksel and M. Skoglund
Stabilization of linear systems over Gaussian networks.
IEEE Trans. on Aut. Control, Submitted June 2012.

Further Extensions...

Gaussian multiple-access, broadcast, and interference channels:

A. Zaidi, T. Oechtering, and M. Skoglund.
Sufficient conditions for closed-Loop control over multiple-access and broadcast channels.
IEEE Int. Conf. on Decision and Control (CDC), 2010

A. Zaidi, T. Oechtering, and M. Skoglund.
Closed-loop stabilization over Gaussian interference channels.
IFAC World Congress, 2011

Summary

Problem: Mean-square stabilization of a discrete-time system over a Gaussian relay network

- Signal-to-noise ratio requirements for stabilization
- Necessary conditions using information theoretic arguments
- Sufficient conditions based on linear delay-free policies
  - linear can be optimal, but not in general
- Connection to achievable rates